

Elliptic Associators and the LMO Functor

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Abstract

The elliptic associator of Enriquez can be used to define an invariant of tangles embedded in the thickened torus, which extends the Kontsevich integral. This construction by Humbert uses the formulation of categories with elliptic structures.

In this work we show that an extension of the LMO functor also leads to an elliptic structure on the category of Jacobi diagrams which is used by the Kontsevich integral, and find the relation between the two structures. We use this relation to give an alternative proof for the properties of the elliptic associator of Enriquez. Those results can lead the way to finding associators for higher genera.

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1 Introduction

The Kontsevich integral was originally defined as an invariant of knots which returns values in a space of Jacobi diagrams ([17]). Its importance comes mainly from the fact that it is universal with respect to Vassiliev invariants (see also [1]). A new, combinatorial, formulation of the Kontsevich invariant for framed knots and links was given in [18]. The idea of this definition is to break the link to elementary tangles and assign values to each of them separately.

The paper [18] already contains the extension of the Kontsevich integral to tangles, which is defined more explicitly in [19] and [3]. In this context it is convenient to consider all tangles as morphisms in the category $q\tilde{T}$ of non-associative framed tangles. This category has the structure of a ribbon category - a concept which encapsulates the above elementary tangles and the relations between them. The Kontsevich integral becomes a functor to a category of Jacobi diagrams \mathcal{A}^∂ , which also gets a structure of a ribbon category.

In [16], the Kontsevich integral is further extended to an invariant of tangles embedded in a thickened torus $\mathbb{T} \times I$. The category of those tangles is denoted $q\tilde{T}_1$, and is an extension of the ribbon category $q\tilde{T}$. The new ingredients in this category are two elementary tangles which go around the generators of $\pi_1(\mathbb{T})$. Those tangles are called beaks. The beaks, together with the relations between them and the other elementary tangles, is encapsulated in the concept of an elliptic structure.

In order to define the extension of the Kontsevich integral to $q\tilde{T}_1$, one needs to find a category of Jacobi diagrams extending \mathcal{A}^∂ such that this extension has an elliptic structure. This category is defined in [16] and denoted \mathcal{A}_1 - the category of elliptic Jacobi diagrams. To give this extension an elliptic structure, [16] uses the concept of an elliptic associator. An elliptic associator is a pair of elements in the exponent of $\hat{f}(A, B)$ - the completed free Lie algebra generated by A and B , satisfying several identities. This associator, when mapped into \mathcal{A}_1 , determines the value of the Kontsevich invariant on the beaks. A specific elliptic associator $e(\phi)$ is introduced in [8] and [13].

In a different direction, the Kontsevich integral was also used to define the LMO invariant, which is an invariant of closed 3-manifolds which returns values in some spaces of Jacobi diagrams ([20]). It was extended to a TQFT first in [21] and, several years later, in [9]. A functorial variant of this construction was given in [10], called the LMO functor.

In this work we extend the LMO functor to an invariant of 3-cobordisms with embedded tangles. Restricted to tangles embedded in the thickened torus, this invariant can be compared to the invariant of [16]. It turns out that those invariants **are not** equal. However, their values on the beaks are equivalent modulo a certain relation called the homotopy relation. We use this equivalence to give an alternative, more intuitive, proof that $e(\phi)$ is indeed an elliptic associator.

This work is divided into the following sections:

In section 2 we review the definitions of ribbon categories and elliptic structures. As we explained, those concepts encapsulate the structure of the categories of tangles $q\tilde{T}$ and $q\tilde{T}_1$, and give us a tool to define invariants of tangles.

In section 3 we review several variants of categories of Jacobi diagrams which will be used later, and explain the relations between them.

In section 4 we define our extension of the LMO functor to the category of embedded tangles in

3-cobordisms. For that purpose we explain how to represent a tangle embedded in a 3-cobordism using a representing tangle in $D^2 \times I$. We also give an explicit description of the beaks using those representing tangles, and verify their properties.

In section 5 we introduce the Lie algebras $\mathfrak{t}_{1,n}$, and explain how their universal enveloping algebras are mapped into the spaces $\mathcal{A}_1^{<p}(\uparrow^n)$ of Jacobi diagrams. For $n = 2, 3$ we prove that a certain restriction of this map is actually an injection into a certain quotient of $\mathcal{A}_1^{<p}(\uparrow^n)$.

In section 6 we introduce the concept of elliptic associators and the elliptic associator $e(\phi)$ of [13]. We prove our main theorem, which determines the relation between $e(\phi)$ and the extended LMO functor. We conclude by using this theorem to give a new proof that $e(\phi)$ is indeed an elliptic associator.

Our results can lead the way to extending the concept of elliptic associators to higher genera, and to finding specific such associators using the extended LMO functor.

2 Categories of Tangles

In this section we define the concepts of ribbon categories and elliptic structures, and introduce the categories of tangles which are the universal examples for those concepts.

2.1 Ribbon Categories

In this subsection we recall the definition of a ribbon category, and give the main example - the category of tangles in $D^2 \times I$. The definitions are all taken from [16].

Let $(\mathcal{C}, \otimes, \mathbf{1}, a)$ be a non-associative monoidal category. Assume for simplicity that $\mathbf{1} \otimes U = U \otimes \mathbf{1} = U$ for any object $U \in \mathcal{C}$. For shortness we will denote $U \otimes V$ by UV . a is a family of natural isomorphisms $a_{X,Y,Z} : (XY)Z \rightarrow X(YZ)$ for any objects $X, Y, Z \in \mathcal{C}$, satisfying the pentagon relation for any objects $X, Y, Z, W \in \mathcal{C}$:

$$a_{W,X,YZ}a_{W,X,Y,Z} = (id_W \otimes a_{X,Y,Z})a_{W,XY,Z}(a_{W,X,Y} \otimes id_Z)$$

A duality on \mathcal{C} is a rule that associates to each object V an object V^* and 2 morphisms $b_V : \mathbf{1} \rightarrow V \otimes V^*$ and $d_V : V^* \otimes V \rightarrow \mathbf{1}$, satisfying:

$$(id_V \otimes d_V)a_{V,V^*,V}(b_V \otimes id_V) = id_V$$

$$(d_V \otimes id_{V^*})a_{V^*,V,V^*}^{-1}(id_{V^*} \otimes b_V) = id_{V^*}$$

A braiding on \mathcal{C} is a family of natural isomorphisms $c_{U,V} : UV \rightarrow VU$ for any 2 objects $U, V \in \mathcal{C}$, satisfying for any objects $U, V, W \in \mathcal{C}$:

$$c_{UV,W} = a_{W,U,V}(c_{U,W} \otimes id_V)a_{U,W,V}^{-1}(id_U \otimes c_{V,W})a_{U,V,W}$$

$$c_{U,VW} = a_{V,W,U}^{-1}(id_V \otimes c_{U,W})a_{V,U,W}(c_{U,V} \otimes id_W)a_{U,V,W}^{-1}$$

By convention we denote $c_{U,V}^{-1} := (c_{V,U})^{-1}$.

A twist on a monoidal category with braiding is a family of natural isomorphisms $\theta_V : V \rightarrow V$ for any object $V \in \mathcal{C}$, satisfying:

$$\theta_{UV} = c_{V,U}c_{U,V}(\theta_U \otimes \theta_V)$$

Definition 2.1. A ribbon category is a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a)$ with duality, braiding and twist as above, satisfying, for any object $V \in \mathcal{C}$:

$$(\theta_V \otimes id_{V^*})b_V = (id_V \otimes \theta_{V^*})b_V$$

The main example for a ribbon category is the category of framed oriented tangles, which we will now describe.

Let $D^2 \subset \mathbb{R}^2$ be the unit disk. Denote by b_n a sequence of some fixed n points in the interior of D^2 (say, the n points uniformly distributed along the segment $(-1, 1)$ of the x axis). Let $m, n \geq 0$

be integers, and ω_s, ω_t non-associative words in the symbols $\{+, -\}$ of lengths m, n , respectively. A **tangle** in $D^2 \times I$ of type (ω_s, ω_t) is an oriented 1-manifold γ embedded in $D^2 \times I$, such that its only boundary points are $\gamma \cap (D^2 \times \{0\}) = b_m \times \{0\}$ and $\gamma \cap (D^2 \times \{1\}) = b_n \times \{1\}$, and such that the orientations of the tangle around the points of $b_m \times \{0\}$ and $b_n \times \{1\}$ correspond to the symbols of ω_s and ω_t (a “+” symbol corresponds to a strand going “up”, i.e. in the positive direction of I , and a “−” symbol corresponds to a strand going “down”). The embedding of γ should be piece-wise smooth and transverse to the horizontal surfaces $D^2 \times \{t\}$ at all but a finite number of points. We also require the tangle to be vertical (i.e. of the form $\{z\} \times I$) near the boundary points.

A **framing** on a tangle γ is the homotopy class relative to the boundary of a non-zero normal vector field on the smooth points of γ , such that the limit of this vector field at the non-smooth points is the same from both sides. The vectors based on the boundary points should all be parametrized as $(0, -1, 0)$. In figures we will use the convention of the blackboard framing.

Definition 2.2. The category $q\tilde{T}$ is the category whose objects are non-associative words in $\{+, -\}$, and whose sets of morphisms $q\tilde{T}(\omega_s, \omega_t)$ are the sets of ambient isotopy classes of oriented framed tangles of type (ω_s, ω_t) .

The concept of a ribbon category is designed to encapsulate the structure of $q\tilde{T}$. More specifically, we have the following proposition, which is easy to verify:

Proposition 2.1. $q\tilde{T}$ is a ribbon category, where we define the dual of $\omega = (\omega_1, \dots, \omega_n)$ to be $\omega^* = (-\omega_n, \dots, -\omega_1)$, and:

$$\begin{aligned}
 a_{\omega_1, \omega_2, \omega_3} &= \begin{array}{c} \omega_1 \quad (\omega_2 \quad \omega_3) \\ \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \\ (\omega_1 \quad \omega_2) \quad \omega_3 \end{array} \\
 b_\omega &= \begin{array}{c} \omega \quad \omega^* \\ \text{diagram of a cup with } \omega \text{ on the left and } \omega^* \text{ on the right} \end{array} \\
 d_\omega &= \begin{array}{c} \text{diagram of a cap with } \omega \text{ on the left and } \omega^* \text{ on the right} \\ \omega \quad \omega^* \end{array} \\
 c_{\omega_1, \omega_2} &= \begin{array}{c} \omega_2 \quad \omega_1 \\ \text{diagram of a crossing with } \omega_2 \text{ on top-left and } \omega_1 \text{ on top-right} \\ \omega_1 \quad \omega_2 \end{array} \\
 \theta_\omega &= \begin{array}{c} \omega \\ \text{diagram of a twist} \\ \omega \end{array}
 \end{aligned}$$

In fact, the category $q\tilde{T}$ is the universal ribbon category, in the sense that there is a unique functor from it to any other ribbon category, preserving most of its properties. A precise formulation and proof of this theorem can be found in [22].

2.2 Elliptic Structure

We will now define the concept of elliptic structure. The definitions in this section are also taken from [16].

Let \mathcal{C} be a ribbon category, \mathcal{C}_1 any category, and $\{\cdot\} : \mathcal{C} \rightarrow \mathcal{C}_1$ a functor.

Definition 2.3. An **elliptic structure** relative to $(\mathcal{C} \rightarrow \mathcal{C}_1)$ is a pair (X, Y) of natural automorphisms of the functor $\{\cdot \otimes \cdot\} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}_1$ (i.e. the composition of the tensor product of \mathcal{C} with the given functor $\{\cdot\}$), satisfying the following identities for any objects $U, V, W \in \mathcal{C}$ (where for $Z = X$ or $Z = Y$ we denote $Z'_{U,V,W} := \{a_{U,V,W}^{-1}\}Z_{U,V,W}\{a_{U,V,W}\}$):

$$X_{UV,W} = X'_{U,V,W}\{c_{V,U} \otimes id_W\}X'_{V,U,W}\{c_{U,V} \otimes id_W\} \quad (2.1)$$

$$Y_{UV,W} = Y'_{U,V,W}\{c_{V,U}^{-1} \otimes id_W\}Y'_{V,U,W}\{c_{U,V}^{-1} \otimes id_W\} \quad (2.2)$$

$$Y_{U,V}X_{U,V}Y_{U,V}^{-1}X_{U,V}^{-1} = \{c_{V,U}c_{U,V}\} \quad (2.3)$$

$$Y'_{U,V,W}\{c_{U,V} \otimes id_W\}X'_{V,U,W}\{c_{U,V} \otimes id_W\} = \{c_{V,U} \otimes id_W\}X'_{V,U,W}\{c_{U,V}^{-1} \otimes id_W\}Y'_{U,V,W} \quad (2.4)$$

We will now describe the main example for a category with an elliptic structure, which is the category of framed oriented tangles in the thickened torus.

Let $\mathbb{T} := S^1 \times S^1$ be the torus. We fix an embedding $D^2 \subset \mathbb{T}$. This embedding also gives us an embedding of all the sets of points b_n into \mathbb{T} . The torus \mathbb{T} (minus an open neighborhood of a point at infinity) is depicted in figure 1. This figure also shows the embedded disk D^2 , and 2 generators x and y of $\pi_1(\mathbb{T})$.

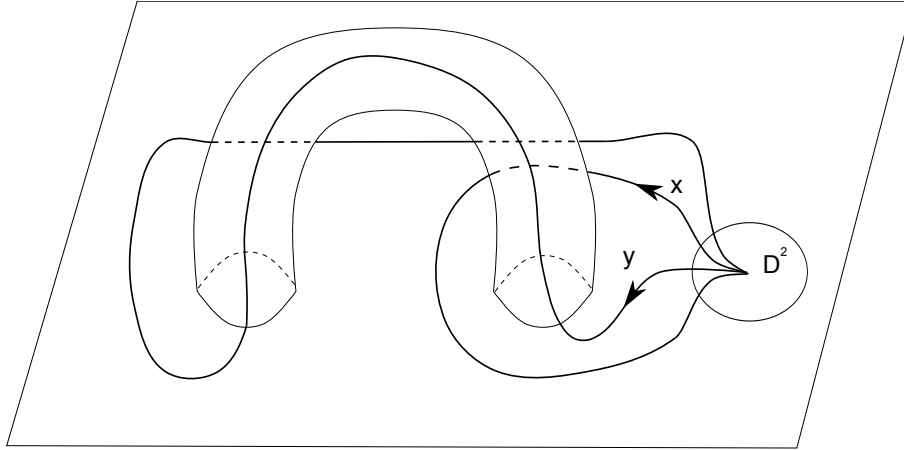


Figure 1: The torus \mathbb{T} with generators of π_1

Definition 2.4. The category $q\tilde{T}_1$ of framed tangles in the thickened torus is defined as follows: The objects of $q\tilde{T}_1$ are non-associative words in $\{+, -\}$. For two such words ω_s, ω_t the morphisms set $q\tilde{T}_1(\omega_s, \omega_t)$ is the set of ambient isotopy classes of oriented framed tangles in $\mathbb{T} \times I$ of type (ω_s, ω_t) . As in the definition of $q\tilde{T}$, the tangles are piece-wise smooth, transverse to the planes $\mathbb{T} \times \{t\}$ at all but a finite number of points, and vertical near the boundary points.

There is an obvious functor $q\tilde{T} \rightarrow q\tilde{T}_1$, induced by the embedding $D^2 \subset \mathbb{T}$. We want to describe an elliptic structure relative to this functor. In this context, the natural automorphisms X_{ω_1, ω_2} and Y_{ω_1, ω_2} act by composition with some invertible tangles in $q\tilde{T}_1((\omega_1)(\omega_2), (\omega_1)(\omega_2))$. We will now describe those tangles.

The tangent space at any point $p = (u, t) \in \mathbb{T} \times I$ can be decomposed as $T_{\mathbb{T}_u} \times \mathbb{R}$. If the point u is in $D^2 \subset \mathbb{T}$, a tangent vector at p can be parametrized by (x, y, t) .

Let $\gamma : [0, 1] \rightarrow \mathbb{T}$ be a smooth simple closed path, with a nowhere vanishing derivative. Assume that $\gamma(0) = \gamma(1)$ is the first (left) point of $b_2 \subset \mathbb{T}$, and that $\gamma'(0) = c_1(-1, 0)$ and $\gamma'(1) = c_2(1, 0)$ for some constants c_1, c_2 (we will assume that the loops representing x and y from figure 1 have this property). We define the tangle $\tilde{\gamma}$ of type $(++, ++)$ as follows:

The right strand is a constant strand at the second (right) point of b_2 , with a framing parametrized constantly by $(0, -1, 0)$.

The left strand is the tangle defined by $(\gamma(t), t) \in \mathbb{T} \times I$, and framed by the unique framing which has the following two properties: (A) it is parametrized by $(0, 1, 0)$ at $(\gamma(0), 0)$ (thus it is actually not in $q\tilde{T}_1(++ , ++)$ as defined above), and (B) its parametrization has $t = 0$ at all points (this parametrization is unique due to the nowhere vanishing derivative assumption).

By following closely the loops of figure 1 it can be seen that at $\tilde{\gamma}(1)$ this framing is parametrized by $(0, -1, 0)$, for both $\gamma = x$ and $\gamma = y$.

Let pt (=positive twist) and nt (=negative twist) be the following tangles of type $(++, ++)$: The right strand is constant, as in the definition of $\tilde{\gamma}$. The left strand is also a constant strand, but its framing makes a half twist from $(0, -1, 0)$ to $(0, 1, 0)$. In pt this twist is in the positive direction, and in nt the twist is in the negative direction.

For any 2 words ω_1, ω_2 and any tangle $u \in q\tilde{T}_1(++ , ++)$, define the cabling $\Delta_{\omega_1, \omega_2}^{++}(u) \in q\tilde{T}_1((\omega_1)(\omega_2), (\omega_1)(\omega_2))$ to be the tangle obtained from u by duplicating the left strand $|\omega_1|$ times along its framing and giving the strands orientations according to the symbols of ω_1 , and similarly for the right strand with ω_2 .

Proposition 2.2. *There is an elliptic structure relative to $(q\tilde{T} \rightarrow q\tilde{T}_1)$ with:*

$$X_{\omega_1, \omega_2} = \Delta_{\omega_1, \omega_2}^{++}(\tilde{x} \cdot pt)$$

$$Y_{\omega_1, \omega_2} = \Delta_{\omega_1, \omega_2}^{++}(\tilde{y} \cdot nt)$$

where x, y are the representatives of the elements of $\pi_1(\mathbb{T})$ depicted in figure 1.

The compositions $\tilde{x} \cdot pt$ and $\tilde{y} \cdot nt$ which appear in this proposition are defined by simply putting the first tangle on top of the second. Note that the framings of \tilde{x} and \tilde{y} at the bottom correspond to the framings of pt and nt at the top, and after the compositions we get elements in $q\tilde{T}_1(++ , ++)$.

This proposition is proved in [16] as a special case of the general concept of genus g structures. Furthermore, it is proved there that $(q\tilde{T} \rightarrow q\tilde{T}_1)$ is universal, in the sense that there is a unique pair of functors from it to any other pair with an elliptic structure, preserving most of its properties.

It is clear that the main ingredients of this elliptic structure are the tangles $\tilde{x} \cdot pt$ and $\tilde{y} \cdot nt$. In [16], these are the tangles which are represented by “beaks” in beak diagrams. In section 4 we

will give a different description of those tangles, and use this description to give another, pictorial, proof of proposition 2.2.

3 Categories of Jacobi Diagrams

In this section we review the definition of several categories of Jacobi diagrams, and the relations between them. Most of those categories are obvious extensions of spaces which appear, in one way or another, in [10], [15] and [16]. In the following sections we will see how those categories are used to define invariants of tangles.

3.1 Category of Patterns

Definition 3.1. A **pattern** P is a compact (but not necessarily closed) oriented 1-manifold whose boundary ∂P is divided into 2 ordered sets $\partial^s P$ and $\partial^t P$. To each pattern P we can associate 2 words $\omega_s(P)$ and $\omega_t(P)$ in the symbols $\{+, -\}$. The words $\omega_s(P)$ and $\omega_t(P)$ encode the orientations of P around the ordered sets $\partial^s P$ and $\partial^t P$: A $+$ ($-$) in the i -th place of $\omega_s(P)$ means that the orientation of P near the i -th point of $\partial^s P$ is going away from (towards) this point. A $+$ ($-$) in the i -th place of $\omega_t(P)$ means that the orientation of P near the i -th point of $\partial^t P$ is going towards (away from) this point.

The category of patterns \mathbb{P} is the category whose objects are finite words in the symbols $\{+, -\}$, and for any 2 such words ω_s, ω_t the morphisms set $\mathbb{P}(\omega_s, \omega_t)$ is the set of patterns P such that $\omega_s(P) = \omega_s$ and $\omega_t(P) = \omega_t$. The composition $P_2 \cdot P_1$ of 2 patterns is defined by attaching $\partial^t P_1$ to $\partial^s P_2$. Graphically, we usually draw $\partial^s P$ at the bottom and $\partial^t P$ at the top. The composition is then obtained by putting P_2 above P_1 .

Note: There are obvious functors $q\tilde{T} \rightarrow \mathbb{P}$ and $q\tilde{T}_1 \rightarrow \mathbb{P}$ which map an object ω to itself (forgetting the non-associative structure), and a tangle u to its skeleton. The split $\partial P = \partial^s P \cup \partial^t P$ is determined by whether the boundary point is in $D^2 \times \{0\}$ or in $D^2 \times \{1\}$.

3.2 Jacobi Diagrams and the Category \mathcal{A}

Let P be a pattern, and S a set. A **Jacobi diagram** D over P and S is a uni-trivalent graph whose univalent vertices are either connected to a point in P or labeled by an element of S . The trivalent vertices are cyclically oriented. When we draw a Jacobi diagram in a 2-dimensional plane, we assume the orientations of the trivalent vertices are always counter-clockwise. An example for a Jacobi diagram is given in figure 2. The pattern P is given by the solid lines, and the uni-trivalent diagram is given by the dashed lines.

The **degree** of a Jacobi diagram D is defined to be the number of trivalent vertices + the number of univalent vertices on P .

The set of all Jacobi diagrams over P and S is denoted $\mathbb{D}(P, S)$.

Notation: Let A be a set with a degree map $A \rightarrow \mathbb{N}$, and \mathbb{F} a field (\mathbb{F} will usually be a field of characteristic 0, such as \mathbb{R} or \mathbb{C}). Then $S_{\mathbb{F}}(A)$ denotes the degree completion of the vector space spanned by A over the field \mathbb{F} .

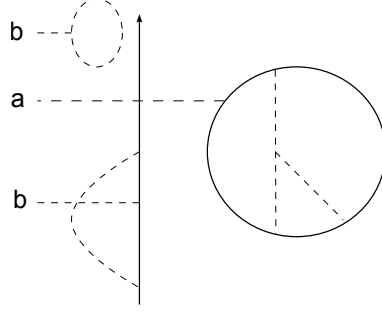


Figure 2: An example for a Jacobi diagram

Let $\mathcal{A}(P, S)$ be the quotient of $S_{\mathbb{P}}(\mathbb{D}(P, S))$ by the relations STU , IHX and AS , which are defined by:

$$\begin{aligned}
 STU : \quad & \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \uparrow \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ \text{---} \end{array} - \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \uparrow \\ \text{---} \end{array} \\
 IHX : \quad & \begin{array}{c} \text{---} \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \text{---} \end{array} = \begin{array}{c} \diagup \\ \text{---} \end{array} \begin{array}{c} \diagdown \\ \text{---} \end{array} - \begin{array}{c} \diagdown \\ \text{---} \end{array} \begin{array}{c} \diagup \\ \text{---} \end{array} \\
 AS : \quad & \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = - \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}
 \end{aligned}$$

At this point we are specifically interested in the spaces $\mathcal{A}(P, \emptyset)$, i.e. spaces of Jacobi diagrams with no labeled vertices. We will denote those spaces by $\mathcal{A}^{\partial}(P)$. In these spaces the degree of all diagrams is even. It is common to define the degree in those spaces as half the number of vertices, but we will continue to use the above definition of degree, in order to be compatible with other spaces of Jacobi diagrams.

Let \mathcal{A}^{∂} be the category defined as follows: The objects of \mathcal{A}^{∂} are words in the symbols $\{+, -\}$. For 2 such words ω_s, ω_t , the morphisms set $\mathcal{A}^{\partial}(\omega_s, \omega_t)$ is defined by: $\mathcal{A}^{\partial}(\omega_s, \omega_t) := \bigcup_{P \in \mathbb{P}(\omega_s, \omega_t)} \mathcal{A}^{\partial}(P)$. For $a \in \mathcal{A}^{\partial}(\omega_s, \omega_t)$ and $b \in \mathcal{A}^{\partial}(\omega_t, \omega_u)$, the composition $b \cdot a$ is obtained by putting b above a and composing the underlying patterns.

Recall the box notation, which is defined in figure 3. In this figure the vertical lines going through the box can be either solid or dashed. The sign ε_i is -1 if the i -th line is a solid line with orientation opposite to the orientation of the box (i.e. the direction which the arrow in the box points to), and $+1$ otherwise.

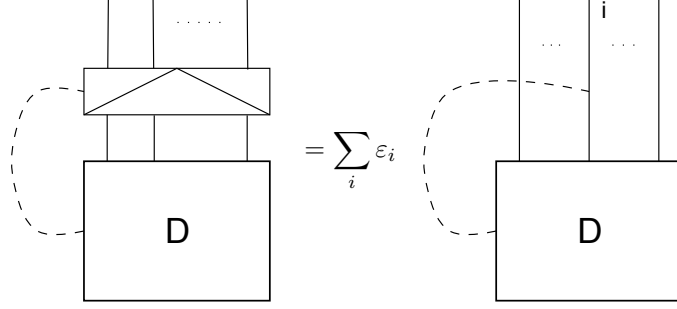


Figure 3: The box notation

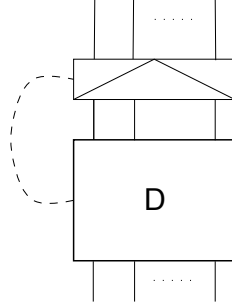


Figure 4: An I relation

In each space $\mathcal{A}^\partial(P)$, we define $\mathbf{I}(P)$ to be the subspace generated by all sums of the type given in figure 4. In this figure we assume that the lines which come out of the upper side of the box are exactly all the ends of the solid lines which lead to the points of $\partial^t(P)$. We call each such sum an I relation.

It can be verified that the union of all $\mathbf{I}(P)$ is a two-sided ideal of \mathcal{A}^∂ (see [16], Lemma 1.4.4). Therefore we can define the quotient category $\mathcal{A}^\partial/\mathbf{I}$. We denote this category by \mathcal{A} .

Remark: As we will see in the following sections, the categories of Jacobi diagrams decorated by ∂ are usually the targets of invariants of tangles in cobordisms whose boundary surfaces have one boundary component. The corresponding quotient categories with no decoration are used when the boundary surfaces have no boundary. Specifically, \mathcal{A}^∂ is used to defined invariants of tangles in $D^2 \times I$, and \mathcal{A} is used to define invariants of tangles in $S^1 \times I$. In the following sections we will be more interested in invariants of tangles in $\mathbb{T} \times I$. The categories \mathcal{A}^∂ and \mathcal{A} will only be used for technical reasons in the way to define those invariants.

We end this section by recalling a known notation (see [10], notation 3.13):

Notation: Let ω be a word of length n in the symbols $\{+, -\}$, and $\omega_1, \dots, \omega_n$ other words in those symbols. The map $\Delta_{\omega_1, \dots, \omega_n}^\omega : \mathcal{A}^\partial(\omega, \omega) \rightarrow \mathcal{A}^\partial(\omega_1 \cdot \dots \cdot \omega_n, \omega_1 \cdot \dots \cdot \omega_n)$ is the map obtained by applying, for each $1 \leq i \leq n$, the doubling map $\Delta : \mathcal{A}^\partial(\uparrow, \uparrow)$ iterated $|\omega_i| - 1$ times on the i -th strand, and by applying the orientation-reversal map S to each new strand whose corresponding symbol in ω_i does not agree with the i -th symbol in ω . This map also induces a map on the

quotient categories: $\Delta_{\omega_1, \dots, \omega_n}^\omega : \mathcal{A}(\omega, \omega) \rightarrow \mathcal{A}(\omega_1 \cdot \dots \cdot \omega_n, \omega_1 \cdot \dots \cdot \omega_n)$

3.3 Unordered Elliptic Jacobi Diagrams

Let $\mathbb{D}_1(P)$ denote the set $\mathbb{D}(P, H_1(\mathbb{T}))$, i.e. the set of Jacobi diagrams over the pattern P with labels coming from the first homology of the torus. Let $\mathcal{A}_1^\partial(P)$ be the quotient of $S_{\mathbb{F}}(\mathbb{D}_1(P))$ by the relations: STU , IHX , AS and multilinearity. The multilinearity relation is defined as follows:

$$(au + bv) \text{---} \boxed{\text{D}} = a \left(u \text{---} \boxed{\text{D}} \right) + b \left(v \text{---} \boxed{\text{D}} \right) \quad \forall a, b \in \mathbb{F} \quad u, v \in H_1(\mathbb{T})$$

The loops x, y from figure 1 induce generators of $H_1(\mathbb{T})$, which we also denote by x, y . Given this basis (or any other basis), it is easy to see that $\mathcal{A}_1^\partial(P)$ is isomorphic to $\mathcal{A}^\partial(P, \{x, y\})$, and the multilinearity relation is no longer needed. In general it is better to work with the definition which allows all the elements of $H_1(\mathbb{T})$ as labels, because this definition does not depend on a choice of a basis, and also because $\mathcal{A}_1^\partial(P)$ defined this way has a natural action of the symplectic group (see [14]). However, for our needs it would be more convenient to consider only diagrams with x and y labels, and forget about multilinearity.

A strut in an elliptic Jacobi diagram $D \in \mathbb{D}(P, \{x, y\})$ is a component in the diagram of the

form $\begin{array}{c} u \\ \vdots \\ v \end{array}$ with $u, v \in \{x, y\}$. D is called **top-substantial** if it has no struts labeled by y on both

vertices. Let ${}^{ts}\mathcal{A}_1^\partial(P) \subset \mathcal{A}_1^\partial(P)$ be the subspace generated by all the top-substantial diagrams. This restriction will allow us to define a composition of elliptic Jacobi diagrams (see also [10], section 3.1).

For 2 elements $D_1 \in {}^{ts}\mathcal{A}_1^\partial(P_1)$ and $D_2 \in {}^{ts}\mathcal{A}_1^\partial(P_2)$ such that P_1 and P_2 are composable, we define the composition $D_2 \cdot D_1$ to be the sum of all diagrams obtained by putting D_2 on top of D_1 and attaching **all** the y -labeled vertices of D_1 to **all** the x -labeled vertices of D_2 (thus, if the number of y -labeled vertices of D_1 is not equal to the number of x -labeled vertices of D_2 , this sum is empty). This composition is extended linearly to a map $\mathcal{A}_1^\partial(P_2) \times \mathcal{A}_1^\partial(P_1) \rightarrow \mathcal{A}_1^\partial(P_2 \cdot P_1)$.

The category ${}^{ts}\mathcal{A}_1^\partial$ is now defined as the category whose objects are words in the symbols $\{+, -\}$, and for any 2 such words ω_s and ω_t , ${}^{ts}\mathcal{A}_1^\partial(\omega_s, \omega_t) := \bigcup_{P \in \mathbb{P}(\omega_s, \omega_t)} {}^{ts}\mathcal{A}_1^\partial(P)$. The composition is defined as above.

The category ${}^{ts}\mathcal{A}_1^\partial$ is a monoidal category. The tensor product of two objects ω_1 and ω_2 is the concatenation $\omega_1\omega_2$, and the tensor product of 2 diagrams D_1 and D_2 is obtained by putting D_1 alongside of D_2 . In ${}^{ts}\mathcal{A}_1^\partial(\emptyset)$ this tensor product induces a multiplication. With respect to this multiplication we can define an exponent. In particular we have the identity elements:

$$id_\omega = \exp \left(\begin{array}{c} y \\ \vdots \\ x \end{array} \right) \otimes \left| \begin{array}{c} \omega \\ \vdots \\ \omega \end{array} \right|$$

Let $\mathbf{I}_1(P)$ be the subspace of ${}^{ts}\mathcal{A}_1^\partial(P)$ generated by sums of the type shown in figure 5. In this figure we assume that the lines which come out of the upper side of the box are exactly all the ends of the solid lines which lead to the points of $\partial^t(P)$, and all the ends of dashed lines ending with the label y . We call each such sum an I_1 relation.

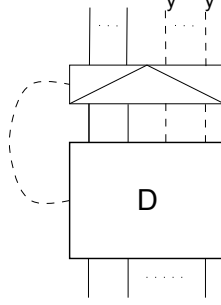


Figure 5: An I_1 relation

It may be verified that the union of all $\mathbf{I}_1(P)$ is a two-sided ideal of ${}^{ts}\mathcal{A}_1^\partial$ (by a similar argument to the one we mentioned in the previous section), which we denote by \mathbf{I}_1 . The quotient category ${}^{ts}\mathcal{A}_1^\partial/\mathbf{I}_1$ is denoted \mathcal{A}_1 . Denote the projection by $\pi : {}^{ts}\mathcal{A}_1^\partial \rightarrow \mathcal{A}_1$.

3.4 Unordered Elliptic Jacobi Diagrams with no Struts

Let $\mathbb{D}_1^y(P) \subset \mathbb{D}_1(P)$ be the subset of all diagrams D such that each component of D has at least one trivalent vertex or one vertex on P . In other words, D has no struts. Since all the relations of ${}^{ts}\mathcal{A}_1^\partial(P)$ preserve this subspace, we get a subspace $\mathcal{A}_1^{\partial y}(P)$ of ${}^{ts}\mathcal{A}_1^\partial(P)$.

We use the spaces $\mathcal{A}_1^{\partial y}(P)$ to define a category $\mathcal{A}_1^{\partial y}$ similarly to the way we defined \mathcal{A}_1^∂ but with a different composition. For 2 elements $D_1 \in \mathcal{A}_1^{\partial y}(P_1)$ and $D_2 \in \mathcal{A}_1^{\partial y}(P_2)$ such that P_1 and P_2 are composable, we define the composition $D_2 \cdot D_1$ to be the sum of all diagrams obtained by putting D_2 on top of D_1 and attaching **some** of the y -labeled vertices of D_1 to **some** of the x -labeled vertices of D_2 . This composition is extended linearly to a map $\mathcal{A}_1^{\partial y}(P_2) \times \mathcal{A}_1^{\partial y}(P_1) \rightarrow \mathcal{A}_1^{\partial y}(P_2 \cdot P_1)$.

Define maps $j_P^\partial : \mathcal{A}_1^{\partial y}(P) \rightarrow {}^{ts}\mathcal{A}_1^\partial(P)$ by $u \mapsto \exp \begin{pmatrix} y \\ \vdots \\ x \end{pmatrix} \otimes u$. It is easy to see that j_P^∂ is

injective. Also, given $u_1 \in \mathcal{A}_1^{\partial y}(P_1)$ and $u_2 \in \mathcal{A}_1^{\partial y}(P_2)$ with P_1 and P_2 composable, it may be verified that $j_{P_2 \cdot P_1}^\partial(u_2 \cdot u_1) = j_{P_2}^\partial(u_2) \cdot j_{P_1}^\partial(u_1)$. So j_P^∂ induce an injective functor $j^\partial : \mathcal{A}_1^{\partial y} \rightarrow {}^{ts}\mathcal{A}_1^\partial$.

Let $\mathbf{I}_1^y(P)$ be the subspace of $\mathcal{A}_1^{\partial y}(P)$ generated by sums of the type shown in figure 6. Again, we assume that the lines which come out of the upper side of the box are exactly all the ends of the solid lines which lead to the points of $\partial^t(P)$, and all the ends of dashed lines ending with the label y . We call each such sum an I_1^y relation.

Proposition 3.1. *The subspaces $\mathbf{I}_1^y(P)$ induce a two-sided ideal of $\mathcal{A}_1^{\partial y}$.*

Proof. Let the two diagrams of figure 6 be denoted by I_{xy} and I_{box} , respectively. Let D_2 be any diagram composable with I_{xy} and I_{box} . We need to show that $D_2 \circ (I_{xy} + I_{\text{box}})$ is in \mathbf{I}_1^y (the other

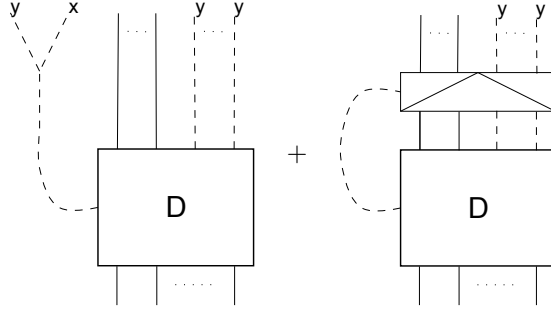


Figure 6: An I_1^y relation

order of composition is trivial).

Denote by A_x the set of all x -labeled vertices of D_2 . Let U_x be a subset of A_x , U_y a subset of the y labels of I_{box} with the same size of U_x , and $p : U_y \xrightarrow{\cong} U_x$ a bijection. Denote by $D_2 \circ_p I_{\text{box}}$ the diagram obtained by attaching the x -labels in U_x to the y -labels in U_y according to p . Similarly define $D_2 \circ_p I_{xy}$. Also, for any $l \in A_x \setminus U_x$, define $D_2 \circ_{(p,l)} I_{xy}$ to be the diagram obtained by attaching the x -labels in U_x to the y -labels in U_y according to p , and also attaching l to the left-most y label in I_{xy} . We have:

$$D_2 \circ (I_{xy} + I_{\text{box}}) = \sum_{(U_y, U_x, p)} \left(D_2 \circ_p I_{\text{box}} + D_2 \circ_p I_{xy} + \sum_{l \in A_x \setminus U_x} D_2 \circ_{(p,l)} I_{xy} \right)$$

For a given triple (U_y, U_x, p) , we will show that the corresponding summand is an I_1^y relation. Indeed, by using IHX and STU relations we can replace the box from I_{box} by a box over all the top solid lines of D_2 , all the edges leading to y labels of D and all the labels leading to the remaining x -labels of D_2 . The summands of this box which are near x -labels get canceled by the sum $\sum_{l \in A_x \setminus U_x} D_2 \circ_{(p,l)} I_{xy}$, and we are left with a new I_1^y relation, as required. \square

We denote the quotient category $\mathcal{A}_1^{\partial y} / \mathbf{I}_1^y$ by \mathcal{A}_1^y , and the projection by $\pi^y : \mathcal{A}_1^{\partial y} \rightarrow \mathcal{A}_1^y$.

For any $u \in \mathbf{I}_1^y(P)$ we have $j_P^\partial(u) \in \mathbf{I}_1(P)$, so the functor j^∂ induces a functor $j : \mathcal{A}_1^y \rightarrow \mathcal{A}_1$.

3.5 Ordered Elliptic Jacobi Diagrams

In this subsection we define the categories $\mathcal{A}_1^{\partial <}$ and $\mathcal{A}_1^{<}$, which are isomorphic to $\mathcal{A}_1^{\partial y}$ and \mathcal{A}_1^y , respectively. Their definition is, in a sense, more complicated - we take more Jacobi diagrams, and quotient them by more relations. On the other hand, the composition rule in these categories is much simpler.

An ordered elliptic Jacobi diagram over a pattern P is an elliptic Jacobi diagram in $\mathbb{D}_1^y(P)$, with the additional data of a linear order on the labeled vertices. Denote the set of all ordered elliptic Jacobi diagrams over P by $\mathbb{D}_1^{<}(P)$. In figures we use the convention that a labeled vertex is bigger if it appears higher in the figure.

Let $\mathcal{A}_1^{\partial <}(P)$ be the quotient of $S_{\mathbb{F}}(\mathbb{D}_1^{<}(P))$ by the relations: STU, AS, IHX, multilinearity

(although we will assume, as above, that the labels are only x, y and there is no need for multilinearity), and STU-like. The STU-like relation is defined as follows:

$$\begin{array}{c} \vdots \\ v \text{ --- } \\ w \text{ --- } \\ \vdots \end{array} \boxed{D} \quad - \quad \begin{array}{c} \vdots \\ w \text{ --- } \\ v \text{ --- } \\ \vdots \end{array} \boxed{D} \quad = \quad \langle v, w \rangle \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \boxed{D}$$

$\langle \cdot, \cdot \rangle$ is the intersection form on $H_1(\mathbb{T})$. Since we will only use the labels x, y , all we need to know is that $\langle y, x \rangle = 1$.

The category $\mathcal{A}_1^{\partial <}$ is defined in a way similar to the previous categories we defined in this section. The composition of 2 diagrams D_1, D_2 is obtained by simply putting D_2 on top of D_1 , and declaring all the labeled vertices of D_2 to be bigger than all the labeled vertices of D_1 . This induces the composition of $\mathcal{A}_1^{\partial <}$ by linearity.

Let $k^\partial : \mathbb{D}_1 \rightarrow \mathbb{D}_1^<$ be the map which sends a diagram D to itself and declares all the x -labeled vertices to be smaller than all the y -labeled vertices. It may be verified that k intertwines the compositions of $\mathcal{A}_1^{\partial y}$ and $\mathcal{A}_1^{\partial <}$, so it induces a functor $k^\partial : \mathcal{A}_1^{\partial y} \rightarrow \mathcal{A}_1^{\partial <}$.

Proposition 3.2. $k^\partial : \mathcal{A}_1^{\partial y} \rightarrow \mathcal{A}_1^{\partial <}$ is an isomorphism.

A short proof of this proposition, using an explicit formula for the inverse of k^∂ , can be found in [10]. We will give here a different proof. The idea of the proof is, for any diagram $D \in \mathbb{D}_1^<(P)$, to iteratively use the STU-like relation to reduce the number of pairs of labeled vertices with $y < x$, until we get a representation of D as a linear combination of diagrams from $\mathbb{D}_1^y(P)$. This simple idea is formalized using the language of filtrations and direct limits. In section 5 we will use this technique several more times.

Proof. Since k^∂ is the identity on objects, it is enough to show that for any pattern P , $k^\partial : \mathcal{A}_1^{\partial y}(P) \rightarrow \mathcal{A}_1^{\partial <}(P)$ is an isomorphism. For that purpose we will construct an inverse map $\varphi^\partial : \mathcal{A}_1^{\partial <}(P) \rightarrow \mathcal{A}_1^{\partial y}(P)$.

For any diagram $D \in \mathbb{D}_1^<(P)$, define $n_{y < x}(D)$ to be the number of pairs of labeled vertices in D which are labeled by x and y , and the x vertex is bigger than the y vertex. Define $(\mathbb{D}_1^<(P))^n := \{D \in \mathbb{D}_1^<(P) \mid n_{y < x}(D) \leq n\}$. The filtration $(\mathbb{D}_1^<(P))^n$ induces a filtration $S_{\mathbb{F}}((\mathbb{D}_1^<(P))^0) \subset S_{\mathbb{F}}((\mathbb{D}_1^<(P))^1) \subset S_{\mathbb{F}}((\mathbb{D}_1^<(P))^2) \subset \dots$.

Denote by $(\mathcal{A}_1^{\partial <}(P))^n$ the quotient of $S_{\mathbb{F}}((\mathbb{D}_1^<(P))^n)$ by the STU, AS, IHX and STU-like relations which are contained in $S_{\mathbb{F}}((\mathbb{D}_1^<(P))^n)$. So the above filtration induces a sequence of maps:

$$(\mathcal{A}_1^{\partial <}(P))^0 \xrightarrow{k_0^\partial} (\mathcal{A}_1^{\partial <}(P))^1 \xrightarrow{k_1^\partial} (\mathcal{A}_1^{\partial <}(P))^2 \longrightarrow \dots$$

$(\mathcal{A}_1^{\partial <}(P))^0$ is isomorphic to $\mathcal{A}_1^{\partial y}(P)$, and the direct limit of the sequence is $\mathcal{A}_1^{\partial <}(P)$. The sequence of maps k_l^∂ induces the map $k^\partial : \mathcal{A}_1^{\partial y}(P) \rightarrow \mathcal{A}_1^{\partial <}(P)$.

For $(l > 0)$, let $\varphi_l^\partial : S_{\mathbb{F}}((\mathbb{D}_1^<(P))^l) \rightarrow S_{\mathbb{F}}((\mathbb{D}_1^<(P))^{l-1})$ be the map defined on a diagram D as follows: If $n_{y < x}(D) < l$, $\varphi_l^\partial(D) = D$. Else, take the highest pair of consecutive vertices labeled $y < x$, and define $\varphi_l^\partial(D)$ by:

$$D := \begin{array}{c} \vdots \\ x \text{ ---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D'} \xrightarrow{\varphi_l^\partial} \begin{array}{c} \vdots \\ y \text{ ---} \\ x \text{ ---} \\ \vdots \end{array} \boxed{D'} - \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \boxed{D'}$$

We claim that φ_l^∂ induces a map $\varphi_l^\partial : (\mathcal{A}_1^{\partial <}(P))^l \rightarrow (\mathcal{A}_1^{\partial <}(P))^{l-1}$. Indeed, if $u \in S_{\mathbb{F}}((\mathbb{D}_1^<(P))^l)$ is an STU, AS or IHX relation, then φ_l^∂ sends u to a sum of corresponding relations in $S_{\mathbb{F}}((\mathbb{D}_1^<(P))^{l-1})$.

Suppose now $u = u_1 + u_2 + u_3 = \begin{array}{c} \vdots \\ v \text{ ---} \\ w \text{ ---} \\ \vdots \end{array} \boxed{D} - \begin{array}{c} \vdots \\ w \text{ ---} \\ v \text{ ---} \\ \vdots \end{array} \boxed{D} - \langle v, w \rangle \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \boxed{D}$ is an STU-like relation. If none of the labels v, w belong to the highest $y < x$ pair in either u_1 or u_2 , then φ_l^∂ sends u to a sum of STU-like relation in $S_{\mathbb{F}}((\mathbb{D}_1^<(P))^{l-1})$. If the pair v, w is the highest $y < x$ pair in u_i ($i = 1$ or $i = 2$), then by definition $\varphi_l^\partial(u) = 0$ if $n_{y < x}(u_i) = l$, and otherwise $\varphi_l^\partial(u) = u$ is again an STU-like relations.

Suppose now that only v or only w belongs to the highest $y < x$ pair, either in u_1 or in u_2 . Assume WLOG that this happens in u_1 , and assume that $v = y$ is the label belonging to the highest $y < x$ pair. so we have:

$$u = u_1 + u_2 + u_3 = \begin{array}{c} \vdots \\ x \text{ ---} \\ y \text{ ---} \\ w \text{ ---} \\ \vdots \end{array} \boxed{D} - \begin{array}{c} \vdots \\ x \text{ ---} \\ w \text{ ---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} - \langle y, w \rangle \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \boxed{D}$$

If $w = x$ then $y < w$ is the highest $y < x$ in u_2 , and we are back to the previous case. If $w = y$ and $n_{y < x}(u_1) < l$, then $\varphi_l^\partial(u) = u$ is again an STU-like relation. Otherwise we have the following calculation:

$$\begin{aligned} \varphi_l^\partial(u) &= \varphi_l^\partial \left(\begin{array}{c} \vdots \\ x \text{ ---} \\ y \text{ ---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} - \begin{array}{c} \vdots \\ x \text{ ---} \\ y \text{ ---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} \right) = \\ &= \begin{array}{c} \vdots \\ y \text{ ---} \\ x \text{ ---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} - \begin{array}{c} \vdots \\ \text{---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} - \begin{array}{c} \vdots \\ y \text{ ---} \\ x \text{ ---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} + \begin{array}{c} \vdots \\ y \text{ ---} \\ \text{---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} \approx \\ &\approx \begin{array}{c} \vdots \\ y \text{ ---} \\ y \text{ ---} \\ x \text{ ---} \\ \vdots \end{array} \boxed{D} - \begin{array}{c} \vdots \\ y \text{ ---} \\ \text{---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} - \begin{array}{c} \vdots \\ \text{---} \\ y \text{ ---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} - \\ &- \begin{array}{c} \vdots \\ y \text{ ---} \\ y \text{ ---} \\ x \text{ ---} \\ \vdots \end{array} \boxed{D} + \begin{array}{c} \vdots \\ \text{---} \\ y \text{ ---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} + \begin{array}{c} \vdots \\ y \text{ ---} \\ \text{---} \\ y \text{ ---} \\ \vdots \end{array} \boxed{D} = 0 \end{aligned}$$

Note that all the diagrams in this calculation are indeed in $S_{\mathbb{F}}((\mathbb{D}_1^<(P))^{l-1})$. The case that $w = x$ is the label belonging to the highest $y < x$ pair in u_1 is similar. This completes the proof that $\varphi_l^\partial : (\mathcal{A}_1^{\partial <}(P))^l \rightarrow (\mathcal{A}_1^{\partial <}(P))^{l-1}$ is well defined.

φ_l^∂ is the inverse of the map $k_{l-1}^\partial : (\mathcal{A}_1^{\partial <}(P))^{l-1} \rightarrow (\mathcal{A}_1^{\partial <}(P))^l$ defined above. Indeed, $\varphi_l^\partial \circ k_{l-1}^\partial = id$ by definition, and $k_{l-1}^\partial \circ \varphi_l^\partial$ sends an element $a \in (\mathcal{A}_1^{\partial <}(P))^l$ to an element equivalent to a by STU-like. Therefore, the family $\{\varphi_l^\partial\}_l$ induces a map $\varphi^\partial : \mathcal{A}_1^{\partial <}(P) \rightarrow \mathcal{A}_1^{\partial y}(P)$ which is the inverse of k^∂ . \square

Let $\mathbf{I}_1^<(P)$ be the subspace of $\mathcal{A}_1^{\partial <}(P)$ generated by sums of the type shown in figure 7. We assume that the lines which come out of the upper side of the box are exactly all the ends of the solid lines which lead to the points of $\partial^t(P)$. We call each such sum an $I_1^<$ relation. The subspaces $\mathbf{I}_1^<(P)$ are together a two-sided ideal of $\mathcal{A}_1^{\partial <}$ ([16], Lemma 2.4.4). We denote the quotient category by $\mathcal{A}_1^<$, and the projection by $\pi^< : \mathcal{A}_1^{\partial <} \rightarrow \mathcal{A}_1^<$.

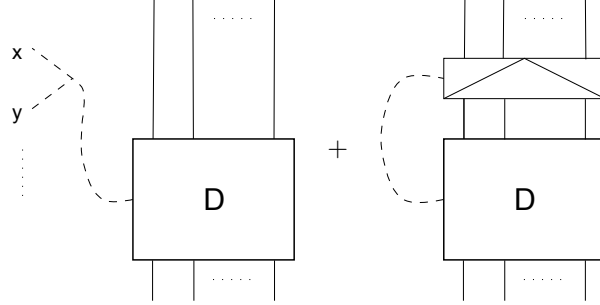


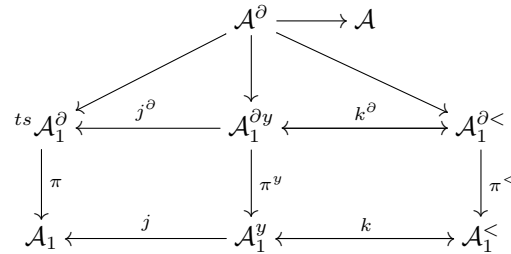
Figure 7: An $I_1^<$ relation

Every I_1^y relation is mapped by k^∂ to an $I_1^<$ relation. Indeed, if we take the I_1^y as shown in figure 6 and apply k^∂ to it, the x label in the first summand will be lower than all the y labels of D . If we now use the STU-like relation to put the x label above the y labels of D , the extra summands we will get in the process will cancel the part of the box in the second summand which is on the y -labeled dashed lines, so we get an $I_1^<$ relation.

The isomorphism k^∂ and its inverse φ^∂ restrict to isomorphisms of the ideals \mathbf{I}_1^y and $\mathbf{I}_1^<$. Therefore we have induced isomorphisms $\mathcal{A}_1^y \xleftarrow[\varphi]{k} \mathcal{A}_1^<$.

3.6 Categories of Pattern-Connected Diagrams

In this section we have defined several categories of Jacobi diagrams. All those categories and the maps between them are summarized in the following diagram. Note that the category \mathcal{A}^∂ has an obvious inclusion into all the categories of elliptic diagrams.



Let D be a Jacobi diagram in any of the sets of Jacobi diagrams defined above. We say that D is **pattern-connected** if all the non-struts components of D have at least one vertex on the pattern. We denote the subsets of pattern-connected Jacobi diagrams by \mathbb{D}^p , \mathbb{D}_1^p , \mathbb{D}_1^{yp} and $\mathbb{D}_1^{<p}$. All the relations we saw respect those subsets, so we can define the corresponding categories of pattern-connected diagrams, which fall into the following diagram:

$$\begin{array}{ccccc}
 & & \mathcal{A}^{\partial p} & \longrightarrow & \mathcal{A}^p \\
 & \swarrow & \downarrow & \searrow & \\
 {}^{ts}\mathcal{A}_1^{\partial p} & \xleftarrow{j^\partial} & \mathcal{A}_1^{\partial yp} & \xleftarrow{k^\partial} & \mathcal{A}_1^{\partial < p} \\
 \downarrow \pi & & \downarrow \pi^y & & \downarrow \pi^< \\
 \mathcal{A}_1^p & \xleftarrow{j} & \mathcal{A}_1^{yp} & \xleftarrow{k} & \mathcal{A}_1^{< p}
 \end{array}$$

Note that in all the categories of pattern-connected Jacobi diagrams we no longer need the AS and IHX relations, because they are implied by STU.

The category $\mathcal{A}^{\partial p}$ is the category which is denoted by \mathbf{A} in [16], and the category $\mathcal{A}_1^{<p}$ is the category which is denoted by \mathbf{A}_1 there. In the following sections we will describe 2 different elliptic structures with respect to $(\mathcal{A}^{\partial p} \rightarrow \mathcal{A}_1^{<p})$.

4 The LMO Functor of Cobordisms with Embedded Tangles

The LMO functor was defined by Cheptea, Habiro and Massuyeau ([10]). It is a functor from the category of Lagrangian cobordisms to a certain category of Jacobi diagrams. In this section we extend the LMO functor to the category of Lagrangian cobordisms with embedded tangles. The extension is quite straight-forward, so most of this section may be seen as a review of CHM's work, with a slight generalization.

On the other hand, our construction will be more restricted than the construction of CHM. They deal with cobordisms between surfaces of any genus, with or without boundary. We will restrict ourselves to closed surfaces of genus 1, which is all we need here. We made this choice for convenience, to make the notation simpler, but the extension to any genus should be obvious.

At the end of this section we show how this extended LMO functor gives rise to an elliptic structure on the categories of Jacobi diagrams introduced in section 3.

4.1 The Category of Lagrangian Cobordisms with Embedded Tangles

Recall that \mathbb{T} is the torus $S^1 \times S^1$. Denote by \mathbb{T}^∂ a torus with one boundary component. A **cobordism of \mathbb{T}^∂** is an oriented compact connected 3-manifold M with an isomorphism $m : \partial(\mathbb{T}^\partial \times [0, 1]) \xrightarrow{\cong} \partial M$. Similarly, a **cobordism of \mathbb{T}** is an oriented compact connected 3-manifold M with an isomorphism $m : \partial(\mathbb{T} \times [0, 1]) \xrightarrow{\cong} \partial M$.

Recall that for any $n \geq 0$ we defined the set of points b_n in \mathbb{T} . We can define those sets of points also in \mathbb{T}^∂ . Let ω_s, ω_t be non-associative words in the symbols $\{+, -\}$, with lengths $|\omega_s| = m, |\omega_t| = n$. A **cobordism with an embedded tangle** of type (ω_s, ω_t) (either of \mathbb{T}^∂ or of \mathbb{T}) is a cobordism M with an embedded framed oriented tangle $T \subset M$ satisfying $\partial T = m((b_m \times \{0\}) \cup (b_n \times \{1\}))$, such that the orientations of T around the boundary points correspond to the words ω_s and ω_t , and the framings around the boundary points are all parallel to the boundary surfaces and parametrized (via m) as $(0, -1, 0)$. As in definition 2.1 above, we require that the tangles be piece-wise smooth and vertical near the boundary points.

Two cobordisms with tangles (M_1, m_1, T_1) and (M_2, m_2, T_2) are said to be equivalent if there is a homeomorphism $h : M_1 \rightarrow M_2$ such that $h \circ m_1 = m_2$ and $h(T_1) = T_2$ (including the framing and the orientation of the tangle).

The categories CT^∂ and CT (Cobordisms with Tangles) are defined as follows: The objects are non-associative words in $\{+, -\}$. For any two such words ω_s, ω_t , the set of morphisms $CT^\partial(\omega_s, \omega_t)$ ($CT(\omega_s, \omega_t)$) is the set of all equivalence classes of cobordisms of \mathbb{T}^∂ (\mathbb{T}) with embedded tangles of type (ω_s, ω_t) . The composition is defined by simply putting one cobordism on top of the other.

We now explain how to represent a cobordism with tangle by another tangle embedded in a simpler manifold. In D^2 we have the sets of points b_n , and we choose 2 more points p and q . Let ω_s, ω_t be non-associative words in $\{+, -\}$. A **representing tangle** of type (ω_s, ω_t) is a framed oriented tangle T embedded in $D^2 \times I$ with the following properties:

- The boundary of T is $((b_{|\omega_s|} \cup \{p, q\}) \times \{0\}) \cup ((b_{|\omega_t|} \cup \{p, q\}) \times \{1\})$.
- There is a component whose boundary is $\{p, q\} \times \{0\}$, denoted by x , and there is a component whose boundary is $\{p, q\} \times \{1\}$, denoted by y .

- The orientations of T around the boundary correspond to the words $(-+)(\omega_s)$ and $(-+)(\omega_t)$. (Note that the convention used in [10] is opposite to ours, so they represent the orientations at p and q by $(+-)$, instead of $(-+)$.)
- We are given a subset S of the closed components of T , and say that those components are marked for surgery.

Two representing tangles T_1 and T_2 are said to be equivalent if they can be related by a sequence of ambient isotopies and Kirby moves on the components marked for surgery. This means that we can add to S a trivial closed component with ± 1 framing or remove such component from S , and we can slide any component over components of S .

The category RT (Representing Tangles) is defined as follows: The objects are non-associative words in $\{+, -\}$. For any two such words ω_s and ω_t , the morphisms set $RT(\omega_s, \omega_t)$ is the set of all equivalence classes of representing tangles of type (ω_s, ω_t) .

There is a simple operation $\circ : RT(\omega_s, \omega_t) \times RT(\omega_t, \omega_u) \rightarrow RT(\omega_s, \omega_u)$ which takes 2 composable tangles and simply puts them one on top of the other. But the definition of the composition in RT is different. Let $T_1 \in RT(\omega_s, \omega_t)$, $T_2 \in RT(\omega_t, \omega_u)$ be representing tangles with subsets marked for surgery S_1 and S_2 , respectively. We define the composition $T_2 \cdot T_1$ to be $T_2 \circ T_i(\omega_t) \circ T_1$, where $T_i(\omega)$ is the tangle shown in figure 8.

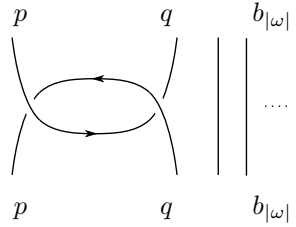
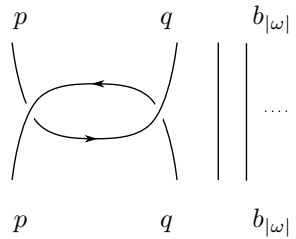
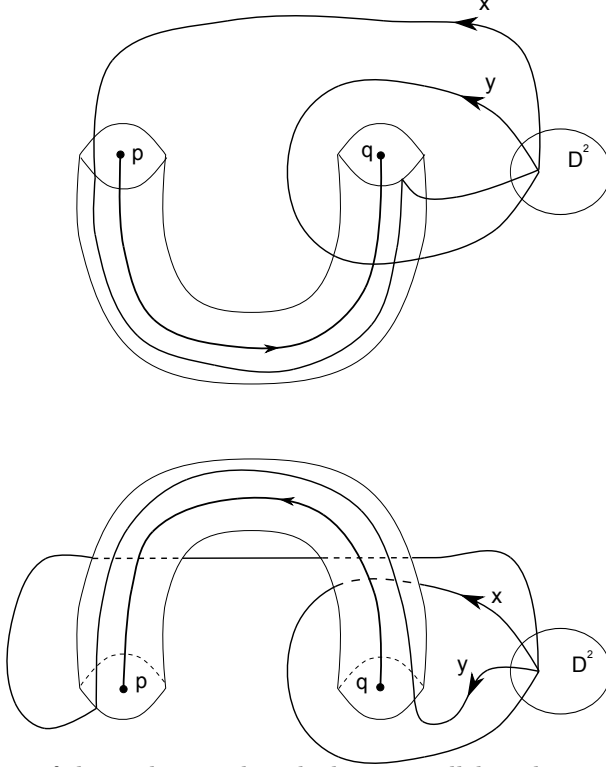


Figure 8: The tangle $T_i(\omega)$

The set of components marked for surgery in $T_2 \cdot T_1$ is defined to be the union of S_1 , S_2 and the (now closed) y component of T_1 and x component of T_2 . The identity in $RT(\omega, \omega)$ is:



There is a functor $rep : RT \rightarrow CT^\partial$, which is the identity on objects, and for a representing tangle T , $rep(T)$ is the cobordism of \mathbb{T}^∂ with embedded tangle obtained by removing a tubular neighborhood of x, y from $D^2 \times I$, and performing surgery on the components in S . The parametrization $m : \partial(\mathbb{T}^\partial \times [0, 1]) \rightarrow \partial rep(T)$ is chosen in such a way that the generators x and y of \mathbb{T}^∂ from figure 1 are mapped to the following elements on the boundary of $rep(T)$:



The segments of the paths x and y which are parallel to the removed components of the tangle are determined by the framing of the tangle.

The functor rep is very much related to the functor D from Theorem 2.10 of [10]. D is a functor from the category of bottom-top tangles in homology cubes to the category of cobordisms. The category of bottom-top tangles in homology cubes, when restricted to tangles with one bottom component and one top component, is isomorphic to the category RT via standard Kirby calculus. Therefore rep factors through D , which proves that it is indeed a functor and an isomorphism.

In order to define the LMO functor we need to restrict to the subcategories $LCT^\partial \subset CT^\partial$ and $LCT \subset CT$ of Lagrangian cobordisms with embedded tangles. The exact definition of Lagrangian cobordisms is not important here, and can be found in [10] (Definition 2.4). For our purposes it is enough to say that the corresponding subcategory LRT of RT is the subcategory of all representing tangles in which the determinant of the linking matrix of S equals ± 1 , and the framing of y , after performing surgery on S , is 0.

4.2 Definition of the LMO Functor

Let T be a tangle in $LRT(\omega_s, \omega_t)$. Denote by $P \in \mathbb{P}(\omega_s, \omega_t)$ the skeleton of T . It is decomposed as $P = \{x, y\} \cup S \cup P'$.

Recall that a Drinfel'd associator is an element $\phi(A, B)$ in the exponent of the completed free Lie algebra generated by A and B , which satisfies several identities (see, for example, [4], Definition 3.1). We define Z to be a functor from the category of tangles to the category of Jacobi diagrams \mathcal{A}^∂ . The tangle T will be mapped by Z to $Z(T) \in \mathcal{A}^\partial(P)$. Z is a variant of the Kontsevich integral of tangles, which is defined over elementary tangles as follows:

$$\begin{aligned}
Z \left(\begin{array}{c} u(vw) \\ \uparrow \uparrow \uparrow \\ (uv)w \end{array} \right) &= \Delta_{uvw}^{+++} \left(\phi \left(\begin{array}{c} \uparrow \cdots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ \uparrow, \uparrow \end{array} \right) \right) \\
Z \left(\begin{array}{c} (++) \\ \nearrow \nwarrow \\ (++) \end{array} \right) &= \exp\left(\frac{1}{2} \begin{array}{c} \nearrow \nwarrow \\ \text{---} \\ \nwarrow \nearrow \end{array} \right) \quad Z \left(\begin{array}{c} (++) \\ \nwarrow \nearrow \\ (++) \end{array} \right) = \exp\left(-\frac{1}{2} \begin{array}{c} \nwarrow \nearrow \\ \text{---} \\ \nearrow \nwarrow \end{array} \right) \\
Z \left(\begin{array}{c} \curvearrowright \\ (- \quad +) \end{array} \right) &= \curvearrowright \quad Z \left(\begin{array}{c} (- \quad +) \\ \curvearrowleft \end{array} \right) = \curvearrowleft
\end{aligned}$$

where $\nu \in \mathcal{A}^\partial(\uparrow) \cong \mathcal{A}^\partial(\bigcirc)$ is the Kontsevich integral of the unknot with 0 framing.

Let $Z^{\nu,S}(T)$ be the value obtained from $Z(T)$ by taking the connected sum of each component of S with ν .

In [10] (after Lemma 4.9), an element $T_g \in \mathcal{A}(\emptyset, \{1^-, \dots, g^-, 1^+, \dots, g^+\})$ is defined. We will consider T_1 as an element of $\mathcal{A}(\emptyset, \{x, y\})$ via the labels change $1^- \mapsto x$ and $1^+ \mapsto y$. For a word ω in the symbols $\{+, -\}$, let id_ω be the identity pattern in $\mathbb{P}(\omega, \omega)$, and let $T_1(\omega) \in {}^{ts}\mathcal{A}_1^\partial(id_\omega)$ be the element obtained by putting T_1 alongside the empty pattern id_ω .

The LMO functor $LMO : LCT^\partial \cong LRT \rightarrow {}^{ts}\mathcal{A}_1^\partial$ is defined as follows: A non-associative word ω is mapped to itself, forgetting the non-associative structure. A morphism $T \in LRT(\omega_s, \omega_t)$ is mapped to:

$$LMO(T) := T_1(\omega_t) \cdot \left(U_+^{-\sigma_+(S)} U_-^{-\sigma_-(S)} \int_S \chi_{S \cup \{x, y\}}^{-1} Z^{\nu,S}(T) \right)$$

where:

- $\chi_{S'} : \mathcal{A}(P', S') \rightarrow \mathcal{A}^\partial(P' \cup S')$ is the symmetrization map defined in [1] (section 5.2, in the proof of Theorem 8), applied to the components of S' (which are first considered as labels). For a diagram $D \in \mathcal{A}(P', S')$, $\chi_{S'}(D)$ is the average of all the diagrams which are obtained by putting all the s labeled vertices on the s component of the pattern S' (for all $s \in S'$), in any possible order.
- $\int_{S'} :$ is the Århus integral on the labels of S' , as defined in [6] (section 2.1, specifically Definition 2.11).
- $U_\pm := \int \chi^{-1}(\nu \# Z(\bigcirc_\pm))$ with \bigcirc_\pm being an unknot with framing ± 1 .
- $\sigma_\pm(S)$ are the numbers of positive/negative eigenvalues of $Lk(S)$.

For a more detailed account of this construction, and a proof of invariance and functoriality, see [10]. For our purposes it is almost enough to use this definition as a “black box”. The only

important thing to notice is that the integral \int_S commutes with $\chi_{\{x,y\}}^{-1}$, i.e. we have:

$$\int_S \chi_{S \cup \{x,y\}}^{-1} Z^{\nu,S}(T) = \chi_{\{x,y\}}^{-1} \int_S \chi_S^{-1} Z^{\nu,s}(T)$$

In order to define the LMO functor on cobordisms of \mathbb{T} with embedded tangles, all we need to do is choose a representing tangle T , map it by LMO to ${}^{ts}\mathcal{A}_1^\partial$, and then map it to \mathcal{A}_1 by the quotient map. Thus we get a functor $LMO : LCT \rightarrow \mathcal{A}_1$. The fact that this functor is well defined is also proved in [10], Theorem 6.2.

4.3 Restrictions of the LMO Functor

Let HCT be the subcategory of LCT containing only tangles in cobordisms which are homology cylinders. Homology cylinders are cobordisms which are homologically equivalent to the cylinder $\mathbb{T} \times I$ (an exact definition can be found in [10] Definition 8.1). The LMO functor, when restricted to

this category, gives values of the form $\exp \left(\begin{pmatrix} y \\ \vdots \\ x \end{pmatrix} \right) \otimes a$, where a is a combination of diagrams from

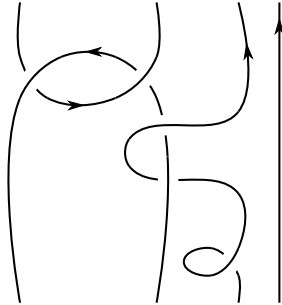
\mathbb{D}_1^y (see [10] section 8.2 and [15] section 4.2). We can therefore compose the LMO functor with the inverse of the injective functor $j : \mathcal{A}_1^y \rightarrow \mathcal{A}_1$ defined above to get a functor $LMO^y : HCT \rightarrow \mathcal{A}_1^y$. Furthermore, we can also define $LMO^< : HCT \rightarrow \mathcal{A}_1^<$ by $LMO^< := k \circ LMO^y$, k being the isomorphism of categories defined in section 3.5.

If we further restrict HCT to include only tangles in the trivial cylinder $\mathbb{T} \times I$, we get the category $q\tilde{T}_1$ of framed tangles in the thickened torus defined in Section 2.2. Restricting the above variants of the LMO functor to this subcategory we get $LMO^y : q\tilde{T}_1 \rightarrow \mathcal{A}_1^{yp}$ and $LMO^< : q\tilde{T}_1 \rightarrow \mathcal{A}_1^{<p}$.

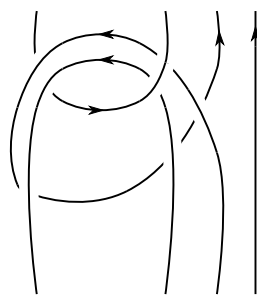
4.4 An Elliptic Structure on $\mathcal{A}_1^{<p}$

In section 2.2 we defined an elliptic structure on $q\tilde{T}_1$. The key ingredients were the tangles $X_{+,+} = \tilde{x} \cdot pt$ and $Y_{+,+} = \tilde{y} \cdot nt$ in $q\tilde{T}_1(++++)$. We will now give an explicit description of those tangles via their representing tangles in $RT(++++)$:

$X_{+,+} =$

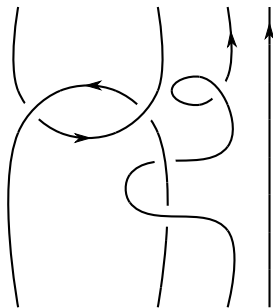


$Y_{+,+} =$



We also have:

$$X_{+,+}^{-1} =$$



$$Y_{+,+}^{-1} =$$



With this representation at hand we can verify the identities (2.1)-(2.4) of definition 2.3 (we verify them only for $U = V = W = +$). In this verification we will use several times the “slam-dunk” move (shown in figure 9), which is implied by the Kirby moves. The equivalences which use this move are marked by a *.

$$(2.1) \quad X_{+,+,+} = \text{Diagram 1} \approx \text{Diagram 2} \approx \text{Diagram 3}$$

Diagrammatic equation (2.1) showing three equivalent knot configurations for $X_{+,+,+}$. The first diagram on the left is a complex knot with multiple crossings and loops. It is followed by an equivalence symbol \approx and a second diagram, which is a slightly rearranged version of the first. This is followed by another \approx and a third diagram, which shows a further simplification or rearrangement of the knot structure. The strands are oriented with arrows throughout.

$$\begin{array}{c}
\approx \\
\begin{array}{c} \text{Diagram 1: A complex knot-like structure with multiple strands and crossings, including a large loop on the left and several smaller loops and crossings in the center. Arrows indicate a flow from top to bottom.} \end{array} \\
\approx^* \\
\begin{array}{c} \text{Diagram 2: A similar complex knot-like structure to Diagram 1, but with different strand configurations and crossings. Arrows indicate a flow from top to bottom.} \end{array} \\
=
\end{array}$$

$$= X'_{+,++}\{c_{+,+} \otimes id_+\}X'_{+,++}\{c_{+,+} \otimes id_+\}$$

$$(2.2) \quad Y_{+,+} = \begin{array}{c} \text{Diagram 3: A complex knot-like structure with multiple strands and crossings, including a large loop on the left and several smaller loops and crossings in the center. Arrows indicate a flow from top to bottom.} \end{array} \approx \begin{array}{c} \text{Diagram 4: A similar complex knot-like structure to Diagram 3, but with different strand configurations and crossings. Arrows indicate a flow from top to bottom.} \end{array} \approx^*$$

$$\begin{array}{c} \approx^* \end{array}
 \begin{array}{c} \text{Diagram 1: A complex knot-like structure with multiple crossings and a vertical line on the right.} \end{array}
 = Y'_{+,++} \{c_{+,+}^{-1} \otimes id_+\} Y'_{+,++} \{c_{+,+}^{-1} \otimes id_+\}$$

$$(2.3) \quad Y_{+,+} X_{+,+} Y_{+,+}^{-1} X_{+,+}^{-1} =$$

$$\begin{array}{c} = \end{array}
 \begin{array}{c} \text{Diagram 2: A complex knot-like structure with multiple crossings and a vertical line on the right.} \end{array}
 \approx^*
 \begin{array}{c} \text{Diagram 3: A complex knot-like structure with multiple crossings and a vertical line on the right.} \end{array}
 \approx^*
 \begin{array}{c} \text{Diagram 4: A complex knot-like structure with multiple crossings and a vertical line on the right.} \end{array}
 \approx
 \begin{array}{c} \text{Diagram 5: A complex knot-like structure with multiple crossings and a vertical line on the right.} \end{array}$$

$$\approx \begin{array}{c} \text{Diagram 1} \end{array} \approx \begin{array}{c} \text{Diagram 2} \end{array} = \{c_{+,+}c_{+,+}\}$$

$$(2.4) \quad \{c_{+,+} \otimes id_+\} X'_{+,++} \{c_{+,+}^{-1} \otimes id_+\} Y'_{+,++} =$$

$$= \begin{array}{c} \text{Diagram 3} \end{array} \approx^* \begin{array}{c} \text{Diagram 4} \end{array} \approx$$

$$\approx \begin{array}{c} \text{Diagram 5} \end{array} \approx^* \begin{array}{c} \text{Diagram 6} \end{array} =$$

$$= Y'_{+,++} \{c_{+,+} \otimes id_+\} X'_{+,++} \{c_{+,+} \otimes id_+\}$$

Applying the maps LMO^y and $LMO^<$ to $X_{+,+}$ and $Y_{+,+}$ we get elliptic structures relative to $\mathcal{A}^\partial \rightarrow \mathcal{A}_1^{yp}$ and $\mathcal{A}^\partial \rightarrow \mathcal{A}_1^{< p}$.

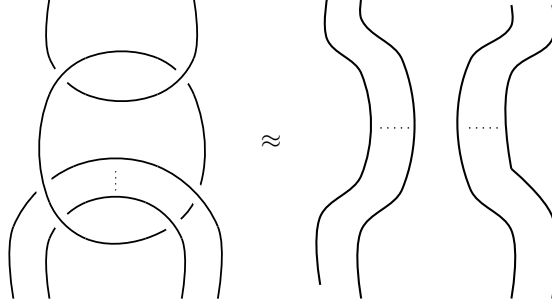


Figure 9: The “slam-dunk” move

5 The Lie Algebras $\mathfrak{t}_{1,n}$ and their Embeddings

In the previous section we saw one way to define an elliptic structure relative to $\mathcal{A}^\partial \rightarrow \mathcal{A}_1^{<p}$, via the LMO functor. Another, more explicit, definition of an elliptic structure comes from specifying certain elements in the algebras $U\hat{\mathfrak{t}}_{1,n}$, and mapping them into $\mathcal{A}_1^{<p}$. In this section we will introduce those algebras and prove some propositions regarding their maps into $\mathcal{A}_1^{<p}$. The actual definition of the elliptic structure will appear in the next section.

5.1 The Lie Algebras $\mathfrak{t}_{1,n}$

Definition 5.1. ([7]) Let $\mathfrak{t}_{1,n}$ be the graded Lie algebra generated by x_i, y_i ($1 \leq i \leq n$) in degree 1 and t_{ij} ($1 \leq i, j \leq n, i \neq j$) in degree 2, with the relations:

$$[v_i, w_j] = \langle v, w \rangle t_{ij}$$

$$[v_i, t_{jk}] = 0$$

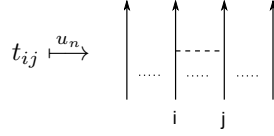
$$[x_i, y_i] = - \sum_{j \neq i} t_{ij}$$

where $1 \leq i, j, k \leq n$ are distinct indices, $v, w \in \{x, y\}$, and $\langle \cdot, \cdot \rangle$ is the intersection form of $H_1(\mathbb{T})$ (the symbols x and y are considered to be the generators of $H_1(\mathbb{T})$ from figure 1). $U\hat{\mathfrak{t}}_{1,n}$ is the degree completion of the universal enveloping algebra of $\mathfrak{t}_{1,n}$.

Denote by \uparrow^n the pattern in $\mathbb{P}(\underbrace{+ \cdots +}_{n \text{ times}}, \underbrace{+ \cdots +}_{n \text{ times}})$ composed of n up-going strands. There is a map $u_n : U\hat{\mathfrak{t}}_{1,n} \rightarrow \mathcal{A}_1^{<p}(\uparrow^n)$ defined by:

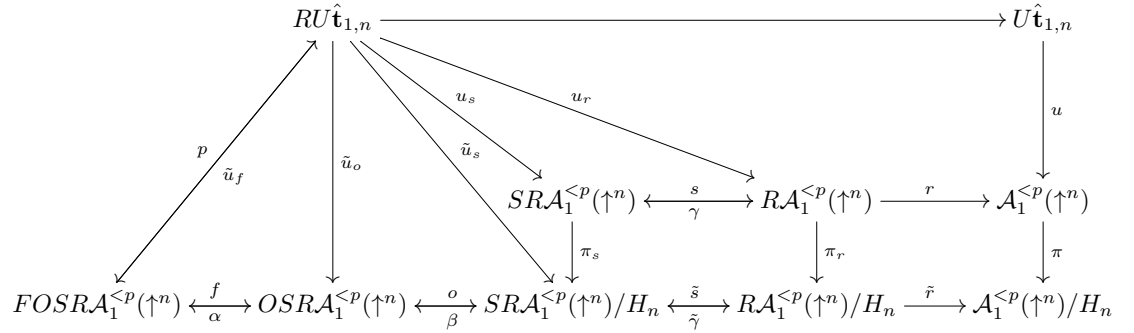
$$v_i \xrightarrow{u_n} \begin{array}{c} \uparrow \\ \vdots \\ v \cdots \cdots \vdots \\ \vdots \\ \vdots \end{array} \quad \begin{array}{c} \uparrow \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}$$

i



It is easily verified that the defining relations of $\mathbf{t}_{1,n}$ are mapped to relations in $\mathcal{A}_1^{<p}(\uparrow^n)$. u_n is an algebra homomorphism, when $\mathcal{A}_1^{<p}(\uparrow^n)$ is considered as an algebra via the composition of $\mathcal{A}_1^{<p}$.

In the next section we will be interested in the subalgebra of $U\hat{\mathbf{t}}_{1,n}$ generated by $\{x_i, y_i \mid 1 \leq i \leq n-1\}$. Denote this subalgebra by $RU\hat{\mathbf{t}}_{1,n}$ (R for “restricted”). We will want to show that for $n=2$ and $n=3$, $u_n|_{RU\hat{\mathbf{t}}_{1,n}}$ is injective. In fact we will prove that for those values of n , u_n is an isomorphism onto a certain quotient of a restriction of $\mathcal{A}_1^{<p}(\uparrow^n)$, as will be explained in section 5.4. In order to prove this theorem, we will define several spaces of Jacobi diagrams which are generated by less diagrams and less relations than $\mathcal{A}_1^{<p}(\uparrow^n)$. The following diagram summarizes the spaces and maps which we will encounter in this section. Note that we should have added a subscript n to all the maps in order to specify the number of strands, but we omit this subscript to make the notations simpler:



Some of the spaces and maps in this diagram are defined for any n , while others are defined only for $n=2$ or $n=3$, as will be clear in the following subsections.

5.2 Restriction to the First $n-1$ Strands

Let $R\mathbb{D}_1^{<p}(\uparrow^n) \subset \mathbb{D}_1^{<p}(\uparrow^n)$ be the subset of all diagrams with no vertices on the rightmost strand. Let $RA_1^{<p}(\uparrow^n)$ be the quotient of $S_{\mathbb{F}}(R\mathbb{D}_1^{<p}(\uparrow^n))$ by all STU and STU-like relations which are contained in $S_{\mathbb{F}}(R\mathbb{D}_1^{<p}(\uparrow^n))$. There is an obvious map $r : RA_1^{<p}(\uparrow^n) \rightarrow A_1^{<p}(\uparrow^n)$. However, it is not a-priori clear that this map is injective, because in $A_1^{<p}(\uparrow^n)$ there are $I_1^{<}$ relations, which relate elements from $R\mathbb{D}_1^{<p}(\uparrow^n)$ to elements outside this subset. The injectivity of r is the goal of this subsection.

Proposition 5.1. $r : RA_1^{<p}(\uparrow^n) \rightarrow A_1^{<p}(\uparrow^n)$ is injective.

We will prove this proposition by representing $A_1^{<p}(\uparrow^n)$ in a different way, which does not involve $I_1^{<}$ relations. However, we start by finding such representation for $A_1^{yp}(\uparrow^n)$, and then we will use the isomorphism $k : A_1^{<p} \rightarrow A_1^{yp}$ to conclude.

Definition 5.2. Let $D \in \mathbb{D}_1^{yp}(\uparrow^n)$ be a diagram, and let v be a vertex in D on the rightmost strand. We call v a **lonely vertex** if it belongs to a component of D which does not have more vertices on the pattern. Otherwise we call it a **non-lonely vertex**.

Let $L\mathbb{D}_1^{yp}(\uparrow^n) \subset \mathbb{D}_1^{yp}(\uparrow^n)$ be the subset of diagrams in which all the vertices on the rightmost strand are lonely vertices. Let $L\mathcal{A}_1^{yp}(\uparrow^n)$ denote the quotient of $S_{\mathbb{F}}(L\mathbb{D}_1^{yp}(\uparrow^n))$ by all STU relations contained in it (there are no I_1^y relations in this subspace).

Proposition 5.2. *The obvious map $l : L\mathcal{A}_1^{yp}(\uparrow^n) \rightarrow \mathcal{A}_1^{yp}(\uparrow^n)$ is an isomorphism.*

Proof. Let $D \in \mathbb{D}_1^{yp}(\uparrow^n)$. For any non-lonely vertex v of D , let $n_l(v)$ be the total degree of all components with a lonely vertex higher than v . Let $n_l(D) := (\sum_{v \text{ non-lonely}} n_l(v)) + |\{v \text{ non-lonely}\}|$. Let $(\mathbb{D}_1^{yp}(\uparrow^n))^m := \{D \in \mathbb{D}_1^{yp}(\uparrow^n) | n_l(D) \leq m\}$. We get a filtration of $\mathbb{D}_1^{yp}(\uparrow^n)$:

$$(\mathbb{D}_1^{yp}(\uparrow^n))^0 \subseteq (\mathbb{D}_1^{yp}(\uparrow^n))^1 \subseteq (\mathbb{D}_1^{yp}(\uparrow^n))^2 \subseteq \dots$$

which induces the sequence:

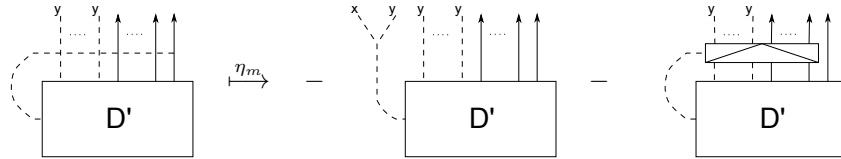
$$S_{\mathbb{F}}((\mathbb{D}_1^{yp}(\uparrow^n))^0) \subseteq S_{\mathbb{F}}((\mathbb{D}_1^{yp}(\uparrow^n))^1) \subseteq S_{\mathbb{F}}((\mathbb{D}_1^{yp}(\uparrow^n))^2) \subseteq \dots$$

Let L^m be the quotient of $S_{\mathbb{F}}((\mathbb{D}_1^{yp}(\uparrow^n))^m)$ by all STU, IHX and I_1^y relations contained in it. We get a sequence:

$$L^0 \xrightarrow{l_0} L^1 \xrightarrow{l_1} L^2 \xrightarrow{l_2} \dots$$

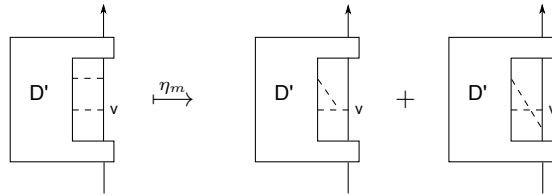
L^0 is isomorphic to $L\mathcal{A}_1^{yp}(\uparrow^n)$ (the IHX relations in this space are implied by STU, and there are no I_1^y relations). The direct limit of this sequence is $\mathcal{A}_1^{yp}(\uparrow^n)$, and the maps l_m induce the map l at the limit. So, in order to complete the proof it is enough to define an inverse for each l_m .

Let $\eta_m : S_{\mathbb{F}}((\mathbb{D}_1^{yp}(\uparrow^n))^m) \rightarrow S_{\mathbb{F}}((\mathbb{D}_1^{yp}(\uparrow^n))^{m-1})$ be defined as follows: For D with $n_l(D) < m$, $\eta_m(D) = D$. For D with $n_l(D) = m$, we have 2 cases. If the highest vertex on the right strand is non-lonely, define $\eta_m(D)$ by:



In these figures we assume that there are no more y labels inside D' .

If the highest vertex on the right strand in D is a lonely vertex, denote the highest non-lonely vertex in D by v , and define $\eta_m(D)$ by:



We claim that η_m induces a map $\eta_m : L^m \rightarrow L^{m-1}$. Indeed, if $u \in S_{\mathbb{F}}((\mathbb{D}_1^{yp}(\uparrow^n))^m)$ is an I_1^y relation then $\eta_m(u)$ is either again an I_1^y relation, or is equal to 0 by definition of η_m . If u is an IHX

relation, $\eta_m(u)$ is a sum of IHX relations. Suppose now u is an STU relation:

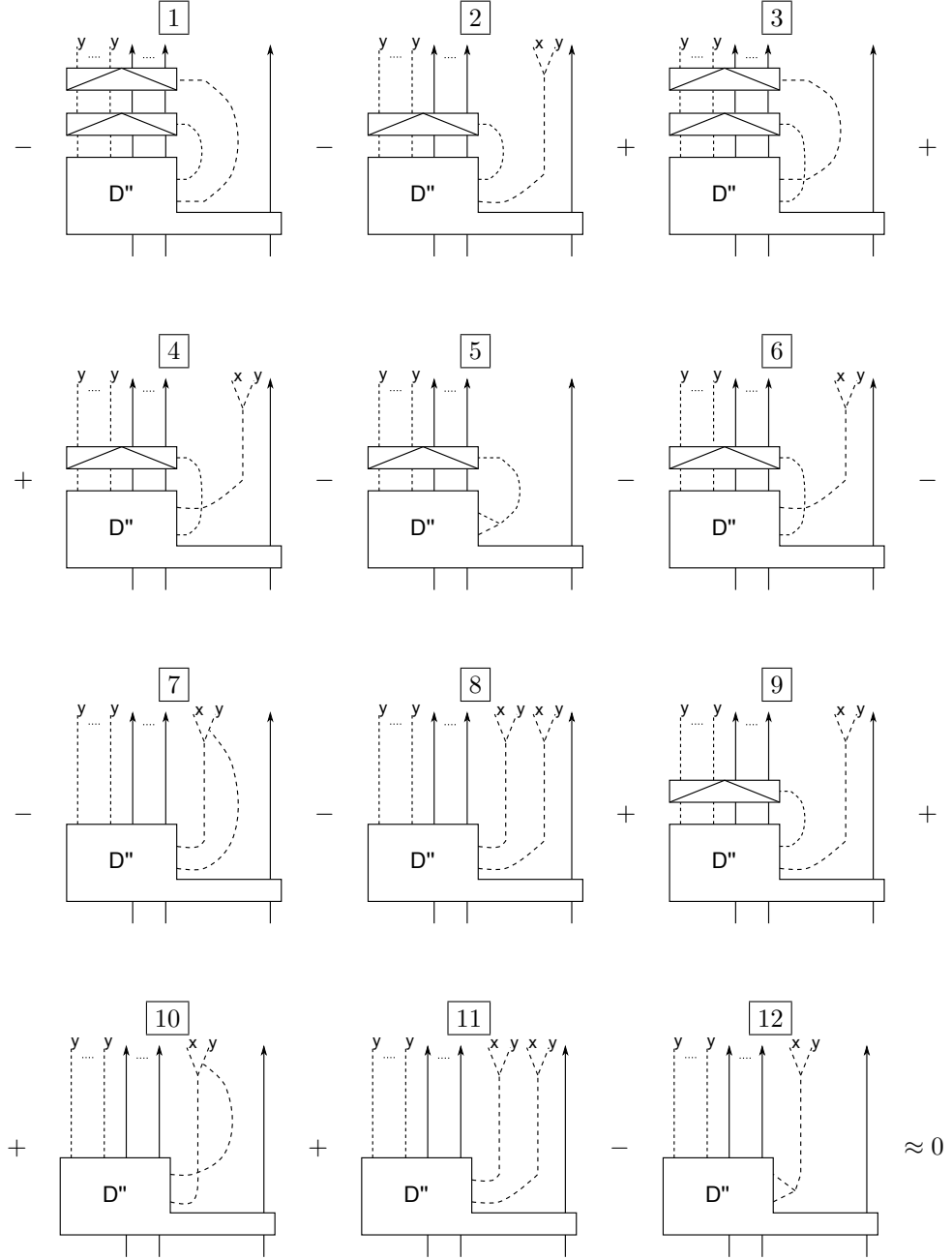
$$u = u_1 - u_2 - u_3 =$$

(Note: the labels v_i , w_i etc. are not part of the diagrams. We write them only to help keep track in the following computations.)

If none of the vertices v_i , w_i are the highest non-lonely vertices in their respective diagrams or the vertices immediately above the highest non-lonely vertices, then $\eta_m(u)$ is a sum of *STU* relations. Otherwise, we have to deal with several different cases (in all those cases, we assume that at least one of u_1, u_2, u_3 has $n_l(u_i) = m$, because otherwise $\eta_m(u) = u$):

- A. If v_3 is the highest non-lonely vertex in u_3 , and either w_1 or w_2 is a lonely vertex, then by definition $\eta_m(u) = 0$.
- B. If v_3 is the highest non-lonely vertex in u_3 and w_1, w_2 are non-lonely (and therefore also v_1 and v_2), then again we have 2 cases:
 - B1. If v_3 is the highest vertex on the right strand, we have:

$$\eta_m(u) \approx$$



We obtain the first equivalence by replacing ① with $\boxed{1}$ and $\boxed{2}$, ② with $\boxed{3}$ and $\boxed{4}$, ④ with $\boxed{6}$, $\boxed{7}$ and $\boxed{8}$, and ⑤ with $\boxed{9}$, $\boxed{10}$ and $\boxed{11}$. The second equivalence follows because: $\boxed{1}$, $\boxed{3}$ and $\boxed{5}$ are a sum of STU and IHX relations, $\boxed{2}$ cancels $\boxed{9}$, $\boxed{4}$ cancels $\boxed{6}$, $\boxed{7}$, $\boxed{10}$ and $\boxed{12}$ are an IHX relation, and $\boxed{8}$ cancels $\boxed{11}$.

B2. If v_3 is not the highest vertex on the right strand, we have:

$$u = u_1 - u_2 - u_3 =$$

and

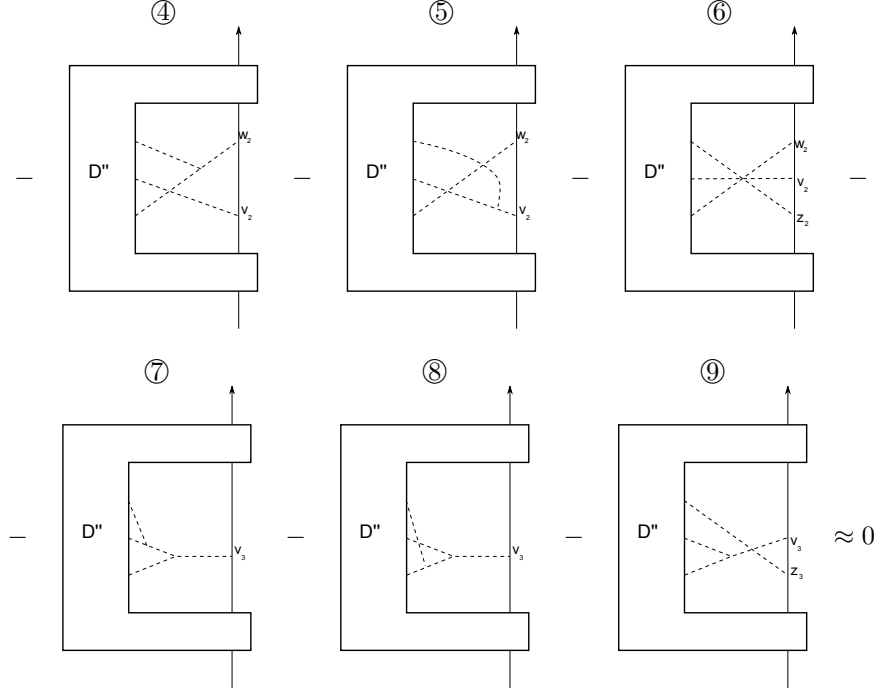
$$\eta_m(u) \approx$$

①

②

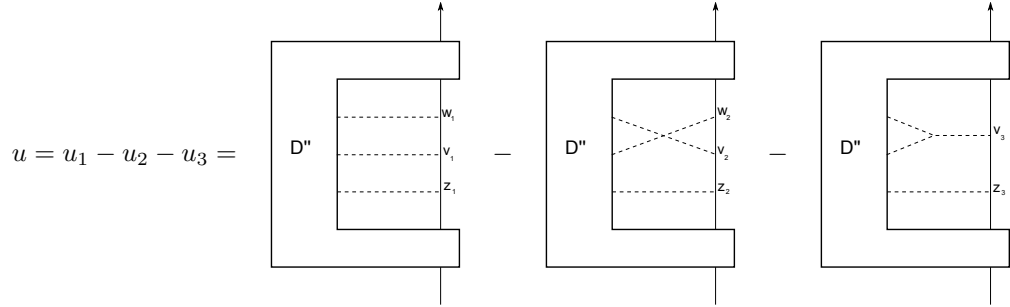
③

$$\approx$$

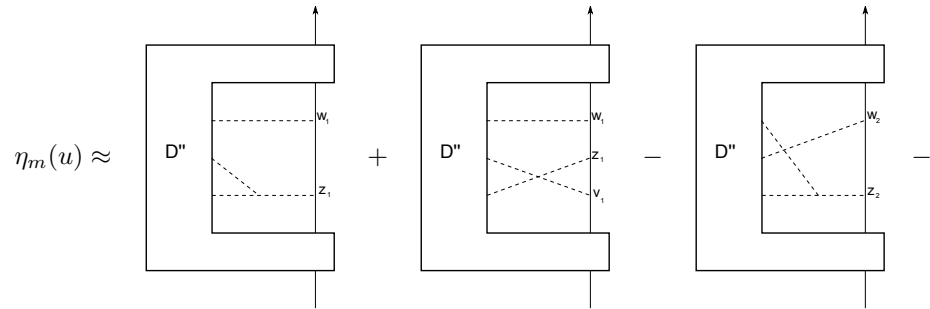


The last equivalence holds because ①, ⑤ and ⑦ are an IHX relation, ②, ④ and ⑧ are an IHX relation and ③, ⑥ and ⑨ are an STU relation.

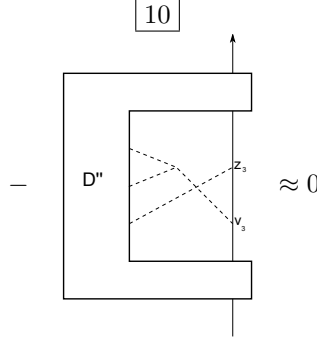
C. If v_3 is the lonely vertex immediately above the highest non-lonely vertex, then:



and



$$\begin{array}{c}
- \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad - \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad - \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad \approx \\
\begin{array}{|c|} \hline \boxed{1} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad + \quad \begin{array}{|c|} \hline \boxed{2} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad + \quad \begin{array}{|c|} \hline \boxed{3} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad + \\
\begin{array}{|c|} \hline \boxed{4} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad - \quad \begin{array}{|c|} \hline \boxed{5} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad - \quad \begin{array}{|c|} \hline \boxed{6} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad - \\
\begin{array}{|c|} \hline \boxed{7} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad - \quad \begin{array}{|c|} \hline \boxed{8} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad - \quad \begin{array}{|c|} \hline \boxed{9} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{D''} \\ \hline \end{array} \quad -
\end{array}$$



The last equivalence holds because $\boxed{1}$, $\boxed{5}$ and $\boxed{9}$ are an IHX relation, $\boxed{2}$ cancels $\boxed{7}$, $\boxed{3}$, cancels $\boxed{6}$, and $\boxed{4}$, $\boxed{8}$ and $\boxed{10}$ are an STU relation.

This completes the proof that $\eta_m : L^m \rightarrow L^{m-1}$ is well defined. It is easy to verify that it is the inverse of l_{m-1} , which completes the proof that l is an isomorphism. \square

We now have a representation of $\mathcal{A}_1^{yp}(\uparrow^n)$ as the quotient of the vector space $S_{\mathbb{F}}(L\mathbb{D}_1^{yp}(\uparrow^n))$ by STU relations alone. Recall the isomorphism $k : \mathcal{A}_1^{yp}(\uparrow^n) \rightarrow \mathcal{A}_1^{<p}(\uparrow^n)$ from section 3. It is easy to see from the proof of proposition 3.2 there that k comes from an isomorphism $k : S_{\mathbb{F}}(\mathbb{D}_1^{yp}(\uparrow^n)) \rightarrow S_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))/\text{STU-like}$. We denote by $LS_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))$ the pre-image in $S_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))$ of $k(S_{\mathbb{F}}(L\mathbb{D}_1^{yp}(\uparrow^n)))$.

Similarly, if $Rel^{yp} \subset S_{\mathbb{F}}(L\mathbb{D}_1^{yp}(\uparrow^n))$ is the subspace generated by all STU relations, denote by $Rel^{<p}$ the pre-image in $S_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))$ of $k(Rel^{yp})$. It is easy to see that $Rel^{<p} \subset LS_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))$ is the subspace generated by all STU and STU-like relations.

Finally we have:

$$\begin{array}{ccc} S_{\mathbb{F}}(L\mathbb{D}_1^{yp}(\uparrow^n))/Rel^{yp} & \xrightarrow[\simeq]{k} & LS_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))/Rel^{<p} \\ \downarrow \simeq & & \downarrow j \\ \mathcal{A}_1^{yp}(\uparrow^n) & \xrightarrow[\simeq]{k} & \mathcal{A}_1^{<p}(\uparrow^n) \end{array}$$

This shows that the obvious map $j : LS_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))/Rel^{<p} \rightarrow \mathcal{A}_1^{<p}(\uparrow^n)$ is actually an isomorphism. Thus we got a presentation of $\mathcal{A}_1^{<p}(\uparrow^n)$ without $I_1^{<}$ relations.

There is no simple description of $LS_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))$ (i.e. it is not a span of a set of diagrams). However, it clearly contains $S_{\mathbb{F}}(R\mathbb{D}_1^{<p}(\uparrow^n))$. Therefore, the map $r : R\mathcal{A}^{<p}(\uparrow^n) \rightarrow \mathcal{A}_1^{<p}(\uparrow^n)$ can be decomposed as:

$$R\mathcal{A}^{<p}(\uparrow^n) \xrightarrow{r'} LS_{\mathbb{F}}(\mathbb{D}_1^{<p}(\uparrow^n))/Rel^{<p} \xrightarrow{j} \mathcal{A}_1^{<p}(\uparrow^n)$$

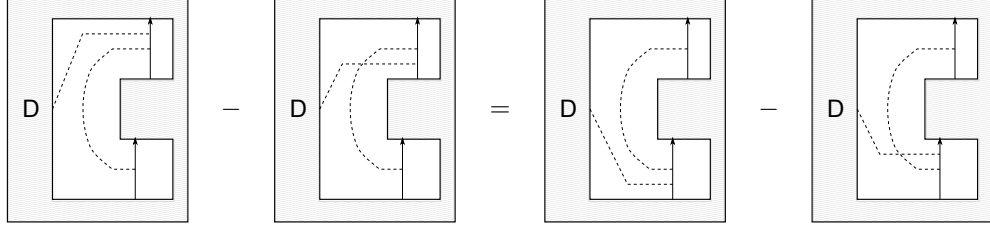
where r' is clearly injective, and therefore r is also injective. This completes the proof of proposition 5.1.

Remark 5.1. If $a \in \mathcal{A}_1^{<p}(\uparrow^n)$ is represented as a sum of diagrams with the property that each component has at least one vertex on one of the first $n - 1$ strands, then a is in the image of r .

Clearly the image $u(RU\hat{\mathbf{t}}_{1,n}) \subset \mathcal{A}_1^{<p}(\uparrow^n)$ is contained in the image $r(R\mathcal{A}_1^{<p}(\uparrow^n)) \subset \mathcal{A}_1^{<p}(\uparrow^n)$. Therefore u induces a map $u_r : RU\hat{\mathbf{t}}_{1,n} \rightarrow R\mathcal{A}_1^{<p}(\uparrow^n)$ satisfying $u = r \circ u_r$. To prove that $u|_{RU\hat{\mathbf{t}}_{1,n}}$ is injective, it is enough to prove that u_r is injective.

5.3 Restriction to Diagrams with no Trivalent Vertices

Let $SR\mathbb{D}_1^{<p}(\uparrow^n)$ (Simple Restricted Diagrams) be the subset of $R\mathbb{D}_1^{<p}(\uparrow^n)$ which contains only the diagrams with no trivalent vertices. Let $SRA_1^{<p}(\uparrow^n)$ be the quotient of $S_{\mathbb{F}}(SR\mathbb{D}_1^{<p}(\uparrow^n))$ by STU-like and 4T relations. The 4T (four terms) relation is defined as follows:



Note that in $R\mathcal{A}_1^{<p}(\uparrow^n)$ the 4T relation is implied by STU.

Proposition 5.3. *The obvious map $s : S_{\mathbb{F}}(SR\mathbb{D}_1^{<p}(\uparrow^n)) \rightarrow S_{\mathbb{F}}(R\mathbb{D}_1^{<p}(\uparrow^n))$ induces an isomorphism $s : SRA_1^{<p}(\uparrow^n) \rightarrow RA_1^{<p}(\uparrow^n)$.*

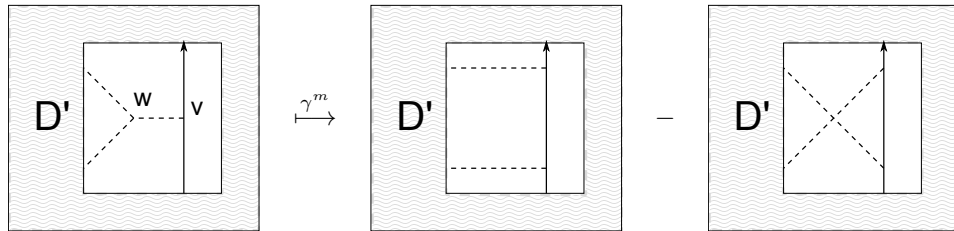
This proposition is a slight generalization of Theorem 6 of [1].

Proof. For a diagram $D \in R\mathbb{D}_1^{<p}(\uparrow^n)$, let $n_{triv}(D)$ be the number of trivalent vertices in D . Let $(R\mathbb{D}_1^{<p}(\uparrow^n))^m$ be the subset of $R\mathbb{D}_1^{<p}(\uparrow^n)$ containing all diagrams D with $n_{triv}(D) \leq m$. We get a filtration of $S_{\mathbb{F}}(R\mathbb{D}_1^{<p}(\uparrow^n))$:

$$S_{\mathbb{F}}((R\mathbb{D}_1^{<p}(\uparrow^n))^0) \subseteq S_{\mathbb{F}}((R\mathbb{D}_1^{<p}(\uparrow^n))^1) \subseteq S_{\mathbb{F}}((R\mathbb{D}_1^{<p}(\uparrow^n))^2) \subset \dots$$

Let S^m be the quotient of $S_{\mathbb{F}}((R\mathbb{D}_1^{<p}(\uparrow^n))^m)$ by STU, STU-like and 4T relations. We get a sequence $S^0 \xrightarrow{s^0} S^1 \xrightarrow{s^1} S^2 \rightarrow \dots$. S^0 is $SRA_1^{<p}(\uparrow^n)$, and the direct limit of the sequence is $RA_1^{<p}(\uparrow^n)$. As in the previous proofs, we need to find an inverse to s^m .

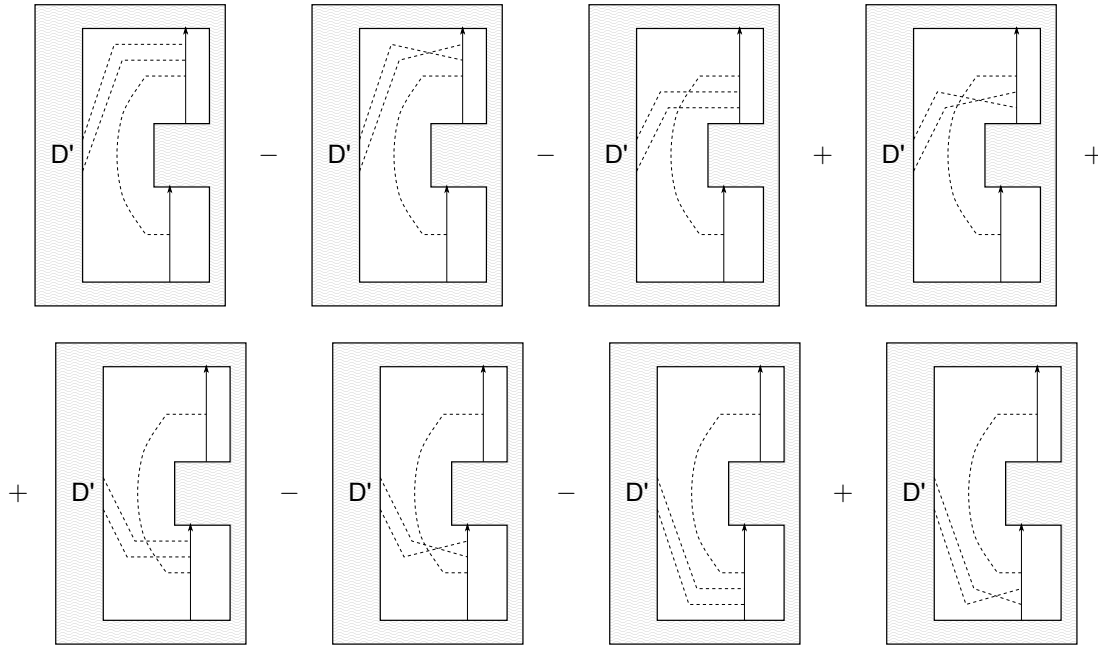
Let $\gamma^m : S_{\mathbb{F}}((R\mathbb{D}_1^{<p}(\uparrow^n))^m) \rightarrow S_{\mathbb{F}}((R\mathbb{D}_1^{<p}(\uparrow^n))^{m-1})$ be defined as follows: Given a diagram $D \in (SR\mathbb{D}_1^{<p}(\uparrow^n))^m$, let $i(D)$ be the left-most strand with a vertex whose component has trivalent vertices. Let $v(D)$ be the highest such vertex on $i(D)$, and let $w(D)$ be the trivalent vertex sharing an edge with $v(D)$. If $n_{triv}(D) < m$, define $\gamma^m(D) = D$. Otherwise, define $\gamma^m(D)$ by:



We claim that γ^m induces a map $\gamma^m : S^m \rightarrow S^{m-1}$. Indeed, if u is an STU-like relation, then $\gamma^m(u)$ is a sum of STU-like relations.

Let $u = u_1 + u_2 + u_3$ be an STU relation, and suppose $n_{triv}(u_1) = m$. If the STU relation u does not involve the vertices $v(u_1)$ and $w(u_1)$, then $\gamma^m(u)$ is a sum of STU relations. If the STU relation involves both $v(u_1)$ and $w(u_1)$, then by definition $\gamma^m(u) = 0$. And if the STU relation involves $w(u_1)$ but not $v(u_1)$, then $\gamma^m(u)$ is a 4T relation.

Now let $u = u_1 + u_2 + u_3 + u_4$ be a 4T relation. If u does not involve, in any of its summands, the vertex $v(u_i)$, then $\gamma^m(u)$ is a sum of 4T relations. If the 4T relation u involves, in any of its summands, the vertex $v(u_i)$, then $\gamma^m(u)$ is either equal to a sum of 4T relations, or equivalent to it via STU (depending on whether $v(u_i)$ is involved in the 4T relation in all the u_i 's or only in 2 of them). This follows from the fact that the following sum:



can be written as a sum of 8 4T relations, with 24 of the 32 summands cancelling in pairs.

It is easy to verify that γ^m is the inverse of s^{m-1} , which completes the proof. \square

The map $u_r : RU\hat{\mathbf{t}}_{1,n} \rightarrow RA_1^{<p}(\uparrow^n)$ induces a map $u_s : RU\hat{\mathbf{t}}_{1,n} \rightarrow SRA_1^{<p}(\uparrow^n)$ by composing with the isomorphism $\gamma : RA_1^{<p}(\uparrow^n) \rightarrow SRA_1^{<p}(\uparrow^n)$. Thus, to prove the injectivity of u_r it is enough to show that u_s is injective.

5.4 Restriction to Ordered Diagrams

From now on we restrict our attention to $n = 2$ and $n = 3$. We wish to show that $u_s : RU\hat{\mathbf{t}}_{1,n} \rightarrow SRA_1^{<p}(\uparrow^n)$ is injective. In fact we will prove that u_s is an isomorphism onto a quotient of $SRA_1^{<p}(\uparrow^n)$. But first we need a definition.

Definition 5.3. A diagram in $SR\mathbb{D}_1^{<p}(\uparrow^n)$ may have 2 kinds of edges: edges with a labeled vertex, which we call **labeled edges**, and edges with both vertices on the pattern, which we call **chords**.

Let $H_n \subset SR\mathbb{D}_1^{<p}(\uparrow^n)$ be the subset of all diagrams with a chord which has 2 vertices on the same strand. (H here stands for “homotopy” - see [2] for an explanation of this notation).

We will denote by the same notation H_n also the image of H_n in $RA_1^{<p}(\uparrow^n)$ via s , and the image in $\mathcal{A}_1^{<p}(\uparrow^n)$ via $r \circ s$. The projections will be denoted by $\pi_s : SRA_1^{<p}(\uparrow^n) \rightarrow SRA_1^{<p}(\uparrow^n)/H_n$, $\pi_r : RA_1^{<p}(\uparrow^n) \rightarrow RA_1^{<p}(\uparrow^n)/H_n$ and $\pi : \mathcal{A}_1^{<p}(\uparrow^n) \rightarrow \mathcal{A}_1^{<p}(\uparrow^n)/H_n$. We also denote $\tilde{u}_s := \pi_s \circ u_s$, $\tilde{u}_r := \pi_r \circ u_r$ and $\tilde{u} := \pi \circ u|_{RU\hat{\mathbf{t}}_{1,n}}$. Most of those spaces and maps can be seen in the diagram in section 5.1.

Remark 5.2. If a diagram $a \in \mathcal{A}_1^{<p}(\uparrow^n)$ has the property described in remark 5.1, and in addition it has a component with more than one vertex on the same strand, or a component with a loop, then a is in the image of H_n via the isomorphism $r \circ s$ (a diagram with a loop is equivalent to a sum of diagrams in H_n after applying the STU relation to all the trivalent vertices).

Theorem 5.4. *The map $\tilde{u}_s : RU\hat{\mathbf{t}}_{1,n} \rightarrow SRA_1^{<p}(\uparrow^n)/H_n$ is an isomorphism for $n = 2, 3$.*

This theorem implies that \tilde{u}_r is also an isomorphism, and \tilde{u} is injective.

In order to prove this theorem we will need to restrict $SR\mathbb{D}_1^{<p}(\uparrow^n)$ further. Let $OSR\mathbb{D}_1^{<p}(\uparrow^n)$ (ordered simple restricted diagrams) be the subset of $SR\mathbb{D}_1^{<p}(\uparrow^n)$ containing all diagrams D which are ordered, in the following sense: If 2 labeled edges have vertices on the same strand, then the order of the labels corresponds to the order of the vertices along the strand. An example for a diagram in $OSR\mathbb{D}_1^{<p}(\uparrow^3)$ is given in figure 10.

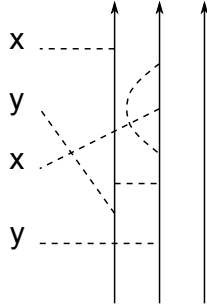
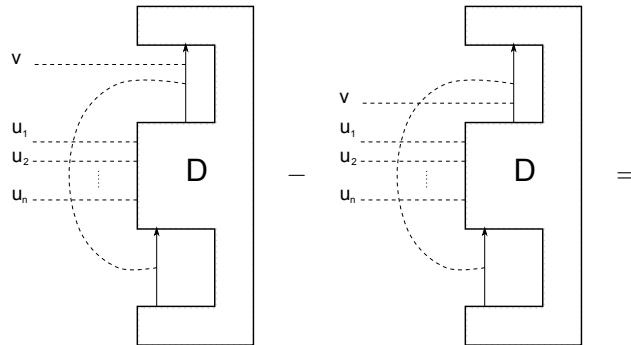
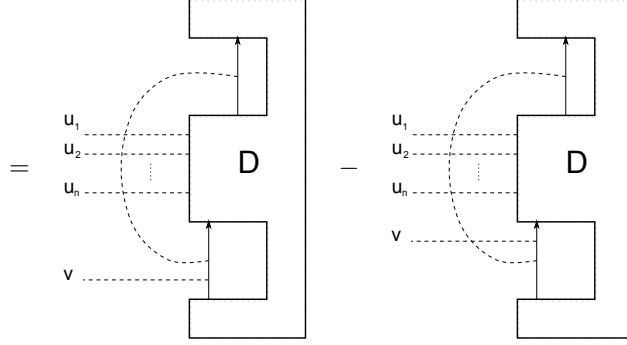


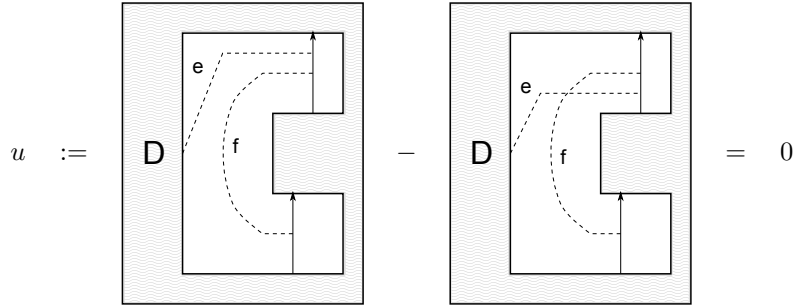
Figure 10: An example for a diagram in $OSR\mathbb{D}_1^{<p}(\uparrow^3)$

We define an O4T (ordered 4 terms) relation to be either a 4T relation or the following relation:





Remark 5.3. In $SRA_1^{<p}(\uparrow^n)/H_n$ ($n = 2, 3$), we have:



if \mathbf{e} is a chord. Indeed, if \mathbf{f} has both vertices on the same strand, then by definition $u \in H_n$. Otherwise, suppose without loss of generality that the upper vertex of \mathbf{f} in the figures is on strand 1, and the lower vertex is on strand 2. If the second vertex of \mathbf{e} is on strand 1, then again $u \in H_n$. Otherwise the other vertex of \mathbf{e} is on strand 2, and u is equivalent via 4T to an element in H_n .

Using this remark we see that in $SRA_1^{<p}(\uparrow^n)/H_n$, O4T is implied by 4T, STU-like and H_n . Let $OSRA_1^{<p}(\uparrow^n)$ be the quotient of $S_{\mathbb{F}}(SR\mathbb{D}_1^{<p}(\uparrow^n))$ by O4T, STU-like and H_n . There is an obvious map $o : OSRA_1^{<p}(\uparrow^n) \rightarrow SRA_1^{<p}(\uparrow^n)/H_n$.

Proposition 5.5. o is an isomorphism.

Proof. Let $D \in SR\mathbb{D}_1^{<p}(\uparrow^n)$, and let e_1 and e_2 be a pair of unordered labeled edges in D . Define $n_o(e_1, e_2) := \#\{\text{labeled vertices between } e_1 \text{ and } e_2\} + 1$, and $n_o(D) := \sum_{e_1, e_2 \text{ unordered in } D} n_o(e_1, e_2)$. n_o induces a filtration:

$$(SR\mathbb{D}_1^{<p}(\uparrow^n))^0 \subset (SR\mathbb{D}_1^{<p}(\uparrow^n))^1 \subset (SR\mathbb{D}_1^{<p}(\uparrow^n))^2 \subset \dots$$

Let O^m be the quotient of $S_{\mathbb{F}}((SR\mathbb{D}_1^{<p}(\uparrow^n))^m)$ by O4T, STU-like and H_n . We get a sequence:

$$O^0 \xrightarrow{o^0} O^1 \xrightarrow{o^1} O^2 \xrightarrow{o^2} \dots$$

O^0 is $OSRA_1^{<p}(\uparrow^n)$, and the direct limit of the sequence is $SRA_1^{<p}(\uparrow^n)/H_n$. As usual, we need to find an inverse to o^m .

Let $\beta^m : S_{\mathbb{F}}((SR\mathbb{D}_1^{<p}(\uparrow^n))^m) \rightarrow S_{\mathbb{F}}((SR\mathbb{D}_1^{<p}(\uparrow^n))^{m-1})$ be defined as follows: if $n_o(D) < m$, $\beta^m(D) = D$. Otherwise, let v be the highest label involved in an unordered pair, and define:

$$\begin{array}{c} v \text{ ---} \\ w \text{ ---} \end{array} \boxed{D'} \xrightarrow{\beta^m} \begin{array}{c} w \text{ ---} \\ v \text{ ---} \end{array} \boxed{D'} + \langle v, u \rangle \begin{array}{c} \text{---} \\ \text{---} \end{array} \boxed{D'}$$

We claim that β^m induces a map $\beta^m : O^m \rightarrow O^{m-1}$. Indeed, if $u \in H_n$ then $\beta^m(u)$ is also in H_n . If u is an STU-like relation:

$$u = u_1 + u_2 + u_3 = \begin{array}{c} v \text{ ---} \\ w \text{ ---} \end{array} \boxed{D'} - \begin{array}{c} w \text{ ---} \\ v \text{ ---} \end{array} \boxed{D'} - \langle v, u \rangle \begin{array}{c} \text{---} \\ \text{---} \end{array} \boxed{D'}$$

there are several cases: If v is the highest label in u_2 involved in an unordered pair, or w is the highest label in u_2 involved in an unordered pair, then by definition $\beta^m(u) = 0$. If the highest label involved in an unordered pair in u_1 (and therefore also in u_2) is z which is immediately above v , then we have:

$$\begin{aligned}
\beta^m(u) &= \beta^m \left(\begin{array}{c} z \text{ ---} \\ v \text{ ---} \\ w \text{ ---} \end{array} \boxed{D''} - \begin{array}{c} z \text{ ---} \\ w \text{ ---} \\ v \text{ ---} \end{array} \boxed{D''} - \langle v, w \rangle \cdot \begin{array}{c} z \text{ ---} \\ \text{---} \end{array} \boxed{D''} \right) \approx \\
&\approx \begin{array}{c} v \text{ ---} \\ z \text{ ---} \\ w \text{ ---} \end{array} \boxed{D''} + \langle z, v \rangle \cdot \begin{array}{c} \text{---} \\ w \text{ ---} \end{array} \boxed{D''} - \begin{array}{c} w \text{ ---} \\ z \text{ ---} \\ v \text{ ---} \end{array} \boxed{D''} - \\
&- \langle z, w \rangle \cdot \begin{array}{c} \text{---} \\ v \text{ ---} \end{array} \boxed{D''} - \langle v, w \rangle \cdot \begin{array}{c} z \text{ ---} \\ \text{---} \end{array} \boxed{D''} \approx \begin{array}{c} \textcircled{1} \\ v \text{ ---} \\ w \text{ ---} \\ z \text{ ---} \end{array} \boxed{D''} + \\
&+ \langle z, w \rangle \cdot \begin{array}{c} \textcircled{2} \\ \text{---} \\ v \text{ ---} \end{array} \boxed{D''} + \langle z, v \rangle \cdot \begin{array}{c} \textcircled{3} \\ \text{---} \\ w \text{ ---} \end{array} \boxed{D''} - \begin{array}{c} \textcircled{4} \\ w \text{ ---} \\ v \text{ ---} \\ z \text{ ---} \end{array} \boxed{D''} - \\
&- \langle z, v \rangle \cdot \begin{array}{c} \textcircled{5} \\ \text{---} \\ w \text{ ---} \end{array} \boxed{D''} - \langle z, w \rangle \cdot \begin{array}{c} \textcircled{6} \\ \text{---} \\ v \text{ ---} \end{array} \boxed{D''} - \\
&- \langle v, w \rangle \cdot \begin{array}{c} \textcircled{7} \\ z \text{ ---} \\ \text{---} \end{array} \boxed{D''} \approx 0
\end{aligned}$$

The last equivalence is true because ② cancels ⑥, ③ cancels ⑤, and ①, ④ and ⑦ are STU-like.

In all other cases, $\beta^m(u)$ is a sum of STU-like relations.

If u is an O4T relation:

$$u = \begin{array}{c} \boxed{\begin{array}{c} \text{D} \\ \text{e} \text{---} \text{f} \end{array}} - \boxed{\begin{array}{c} \text{D} \\ \text{e} \text{---} \text{f} \end{array}} + \\ + \boxed{\begin{array}{c} \text{D} \\ \text{f} \text{---} \text{e} \end{array}} - \boxed{\begin{array}{c} \text{D} \\ \text{f} \text{---} \text{e} \end{array}} \end{array}$$

then again we have to deal with several cases: If \mathbf{e} is a chord, then $\beta^m(u)$ is a sum of O4T relations. Similarly, if \mathbf{e} is a labeled edge, and its label is not “touched” by β^m in either of the summands of u , then again $\beta^m(u)$ is a sum of O4T relations. And if \mathbf{e} is labeled and its label is touched by β^m in some (or all) of the summands of u , then $\beta^m(u)$ is an O4T relation + some terms which are equivalent to 0 according to remark 5.3.

It is easy to see that β^m is the inverse of o^{m-1} , which completes the proof. \square

The map $\tilde{u}_s : RU\hat{\mathbf{t}}_{1,3} \rightarrow SRA_1^{<p}(\uparrow^n)/H_3$ induces a map $\tilde{u}_o : RU\hat{\mathbf{t}}_{1,3} \rightarrow OSRA_1^{<p}(\uparrow^3)$ by composing with the isomorphism $\beta : SRA_1^{<p}(\uparrow^n)/H_3 \rightarrow OSRA_1^{<p}(\uparrow^3)$. Thus, in order to prove theorem 5.4 it is enough to prove that \tilde{u}_o is an isomorphism.

For $n = 2$ there are no relations in $OSRA_1^{<p}(\uparrow^n)$, thus it is the free algebra generated by $\tilde{u}_o(x_1)$ and $\tilde{u}_o(y_1)$. $RU\hat{\mathbf{t}}_{1,2}$ itself is the free algebra generated by x_1 and y_1 . Therefore, we have completed the proof of theorem 5.4 for $n = 2$. For $n = 3$ we will need yet another restriction, which is the content of the next (and final) subsection.

5.5 Restriction to Fully Ordered Diagrams

We begin with some notations:

In a diagram $D \in OSRD_1^{<p}(\uparrow^3)$, each component is an edge. We number the strands from left to right. An edge with one labeled vertex and the other vertex on strand i will be called an i labeled edge. An edge with one vertex on strand i and the other vertex on edge j will be called an i - j edge. Note that we will only have $i = 1$ or $i = 2$.

Let $FOSRD_1^{<p}(\uparrow^3) \subset OSRD_1^{<p}(\uparrow^3)$ (fully ordered simple restricted diagrams) be the union of H_3 with the subset of all diagrams with the following property: the labels of all 1 labeled edges are smaller than the labels of all the 2 labeled edges, and the vertices of all 1 labeled edges on strand

1 are lower than the vertices on strand 1 of all 1-2 edges. Denote by $FOSRA_1^{<p}(\uparrow^3)$ the quotient of $FOSRD_1^{<p}(\uparrow^3)$ by STU-like, O4T and H_3 .

Proposition 5.6. *The obvious map $f : FOSRA_1^{<p}(\uparrow^3) \rightarrow OSRA_1^{<p}(\uparrow^3)$ is an isomorphism.*

Note: There is no obvious multiplication in $FOSRA_1^{<p}(\uparrow^3)$. The content of the proposition is that f is an isomorphism of vector spaces. After we show that f is indeed an isomorphism, it will induce a multiplication on $FOSRA_1^{<p}(\uparrow^3)$ by pulling back the multiplication of $OSRA_1^{<p}(\uparrow^3)$.

Proof. Let $D \in OSRD_1^{<p}(\uparrow^3)$ be a diagram not in H_3 . A pair of edges e_1, e_2 is an unordered pair if e_1 is a 1 labeled edge and e_2 is either a 2 labeled edge with a smaller label or a 1-2 edge with a lower vertex on strand 1. In the first case define $n_f(e_1, e_2) := \#\{\text{labeled vertices between } e_1 \text{ and } e_2\} + 1$, and in the second case define $n_f(e_1, e_2) := \#\{\text{vertices on strand 1 between } e_1 \text{ and } e_2\} + 1$. Define $n_f(D) := \sum_{e_1, e_2 \text{ unordered}} n_f(e_1, e_2)$.

Let $(OSRD_1^{<p}(\uparrow^3))^m$ be the union of H_3 and all diagrams D with $n_f(D) \leq m$. Let F^m be the quotient of $S_{\mathbb{F}}((OSRD_1^{<p}(\uparrow^3))^m)$ by STU-like, O4T and H_3 . We get a sequence:

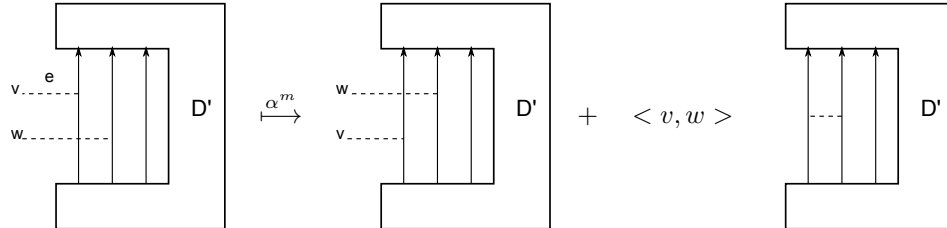
$$F^0 \xrightarrow{f^0} F^1 \xrightarrow{f^1} F^2 \xrightarrow{f^2} \dots$$

The direct limit of the sequence is $OSRA_1^{<p}(\uparrow^3)$, and F^0 is isomorphic to $FOSRA_1^{<p}(\uparrow^3)$. Therefore, what we need to do, as usual, is to find an inverse to f^m .

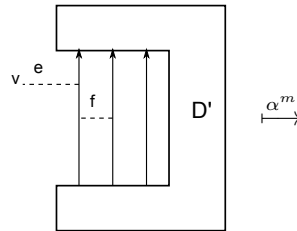
Let $\alpha^m : S_{\mathbb{F}}((OSRD_1^{<p}(\uparrow^3))^m) \rightarrow S_{\mathbb{F}}((OSRD_1^{<p}(\uparrow^3))^{m-1})$ be defined as follows: If $D \in H_3$ or $n_f(D) < m$, $\alpha^m(D) = D$. Else, find the highest 1 labeled edge in D such that either of the following holds:

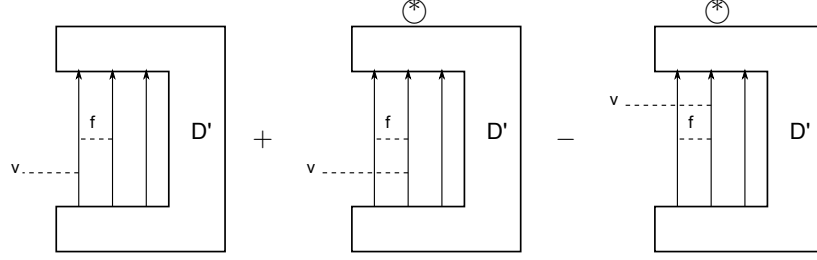
- A. The labeled vertex immediately below it belongs to a 2 labeled edge.
- B. The vertex immediately below it on strand 1 belongs to a 1-2 edge.

Denote this edge by e . If A. applies to e , define:



If only B. applies to e , define:





In the last 2 summands (marked by \circledast) we should specify the location of the label v in the linear order of the labels in D . This is determined as follows: If there is a 2 labeled edge above the vertex of f , locate v as the label immediately below it. Otherwise, locate v as the highest vertex. This choice guaranties that $\alpha^m(D)$ is indeed in $OSRA_1^{<p}(\uparrow^3)$.

We claim that α^m induces a map $\alpha^m : F^m \rightarrow F^{m-1}$. Indeed, assume u is an STU-like relation. Since we are in $OSRA_1^{<p}(\uparrow^3)$, we must have:

$$u = u_1 + u_2 + u_3 =$$

$$= \text{Diagram 1} - \text{Diagram 2} - \langle v, w \rangle \text{Diagram 3}$$

We have $n_f(u_1) > n_f(u_2), n_f(u_3)$. Assume $n_f(u_1) = m$. If v is the highest label with properties A. or B. then by definition $\alpha^m(u) = 0$. Otherwise $\alpha^m(u)$ is equivalent to a sum of STU-like relations.

Assume now that u is an O4T relation:

$$u = u_1 + u_2 + u_3 + u_4 =$$

$$= \text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} - \text{Diagram 4}$$

Clearly we have $n_f(u_1) > n_f(u_2)$ and $n_f(u_3) > n_f(u_4)$. Therefore, potentially we might have

$n_f(u_i) = m$ only for $i = 1$ and $i = 3$. Assume first that only $n_f(u_1) = m$. If v is not the highest label in u_1 with properties A. or B., then $\alpha^m(u)$ is equivalent to an O4T relation. If v is the highest such label and only property B. applies to it, then by definition $\alpha^m(u) = 0$. And if property A. also applies to it, then we have:

$$\begin{aligned}
u = & \text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} - \\
& - \text{Diagram 4} \xrightarrow{\alpha^m} \text{Diagram 5} + \langle v, w \rangle \text{Diagram 6} - \\
& - \text{Diagram 7} - \langle v, w \rangle \text{Diagram 8} + \text{Diagram 9} - \\
& - \text{Diagram 10} \approx 0
\end{aligned}$$

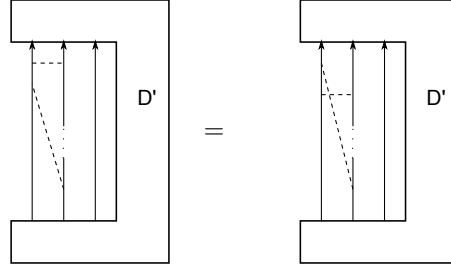
The last equivalence is true because ①, ③, ⑤ and ⑥ are an O4T relation, and ② and ④ are equivalent to 0 by remark 5.3. Note that in some of the diagrams involved in this calculation we might actually have a different order of the labels or a different order along strand 2, but this does not affect the actual calculation.

If only $n_f(u_3) = m$ the argument is similar. If $n_f(u_1) = n_f(u_3) = m$ then the computation is more complicated, since we need to consider several possibilities, but the principles of the calculation are the same, and we leave it to the reader.

It is easy to see that α^m is the inverse of f^{m-1} , thus we have completed the proof. \square

The map $\tilde{u}_o : RU\hat{\mathbf{t}}_{1,n} \rightarrow OSRA_1^{<p}(\uparrow^n)$ induces a map $\tilde{u}_f : RU\hat{\mathbf{t}}_{1,n} \rightarrow FOSRA_1^{<p}(\uparrow^n)$ by composing with the isomorphism $\alpha : OSRA_1^{<p}(\uparrow^n) \rightarrow FOSRA_1^{<p}(\uparrow^n)$. Thus, in order to prove theorem 5.4 it is enough to prove that \tilde{u}_f is an isomorphism.

In $S_{\mathbb{F}}(FOSRD_1^{<p}(\uparrow^3))$ there are no STU-like relations. The only O4T relations involve two 1-2 edges, and modulo H_3 they reduce to the relation:



According to this observation we may further restrict $FOSRD_1^{<p}(\uparrow^3)$. Let $\widetilde{FOSRD_1^{<p}(\uparrow^3)}$ be the subset of $FOSRD_1^{<p}(\uparrow^3)$ containing all diagrams in which the 1 labeled edges are lower than any other edge (as before), and the vertices of all the 1-2 edges have the same order on both strands. $FOSRA_1^{<p}(\uparrow^3)$ will then be isomorphic to the span of $\widetilde{FOSRD_1^{<p}(\uparrow^3)} \cup H_3$ quotiented by H_3 , which is simply $S_{\mathbb{F}}(\widetilde{FOSRD_1^{<p}(\uparrow^3)})$. We denote this space by $\widetilde{FOSRA_1^{<p}(\uparrow^3)}$.

Each diagram in $\widetilde{FOSRA_1^{<p}(\uparrow^3)}$ is a product of the elements $\tilde{u}_f(x_i)$, $\tilde{u}_f(y_i)$ ($i = 1, 2$) and $\tilde{u}_f(t_{12})$. Therefore, there is an obvious map $\widetilde{FOSRA_1^{<p}(\uparrow^3)} \rightarrow RU\hat{\mathbf{t}}_{1,3}$. We denote by p the composition $p : \widetilde{FOSRA_1^{<p}(\uparrow^3)} \xrightarrow{\cong} \widetilde{FOSRA_1^{<p}(\uparrow^3)} \rightarrow RU\hat{\mathbf{t}}_{1,3}$. We claim that p is an inverse to \tilde{u}_f .

Indeed, $\tilde{u}_f \circ p$ is clearly the identity. As for $p \circ \tilde{u}_f$, we need to show that for any $D \in RU\hat{\mathbf{t}}_{1,3}$, the image $p \circ \alpha \circ \tilde{u}_o(D)$ in $RU\hat{\mathbf{t}}_{1,3}$ is equivalent to D via the relations of $U\hat{\mathbf{t}}_{1,3}$. The only map in this composition which actually changes the underlying diagram of D is α . Following the definition of the maps α^m (which compose α) shows that all we need is to verify the following relations in $U\hat{\mathbf{t}}_{1,3}$:

$$-[v_1, t_{12}] = [v_2, t_{12}]$$

$$[v_1, w_2] = < v, w > t_{12}$$

Those relations indeed hold in $U\hat{\mathbf{t}}_{1,3}$ (see [16] Definition 2.1.1 and Lemma 2.1.2). This completes the proof of theorem 5.4.

6 Elliptic Associators and the LMO Functor

In this section we introduce the concept of elliptic associators and the specific associator defined in [8] and [13]. We then study the relation between this elliptic associator and the elliptic structure relative to $\mathcal{A}^\partial \rightarrow \mathcal{A}_1^{<p}$ induced by the LMO functor, which we described in section 4.

6.1 Elliptic Associators

Definition 6.1. ([16]) Let $\hat{f}(A, B)$ be the completed free Lie algebra generated by A and B . Let $\phi \in \exp(\hat{f}(A, B))$ be a Drinfel'd associator. A pair $X(A, B), Y(A, B) \in \exp(\hat{f}(A, B))$ is called **an elliptic associator** with respect to ϕ if it satisfies the following identity in $U\hat{\mathfrak{t}}_{1,2}$:

$$Y(x_1, y_1)X(x_1, y_1)Y^{-1}(x_1, y_1)X^{-1}(x_1, y_1) = \exp(t_{12}) \quad (6.1)$$

and the following 3 identities in $U\hat{\mathfrak{t}}_{1,3}$:

$$\begin{aligned} X(x_1 + x_2, y_1 + y_2) = & \\ & \phi(t_{12}, t_{23})^{-1}X(x_1, y_1)\phi(t_{12}, t_{23})\exp(t_{12}/2) \cdot \\ & \cdot \phi(t_{12}, t_{13})^{-1}X(x_2, y_2)\phi(t_{12}, t_{13})\exp(t_{12}/2) \end{aligned} \quad (6.2)$$

$$\begin{aligned} Y(x_1 + x_2, y_1 + y_2) = & \\ & \phi(t_{12}, t_{23})^{-1}Y(x_1, y_1)\phi(t_{12}, t_{23})\exp(-t_{12}/2) \cdot \\ & \cdot \phi(t_{12}, t_{13})^{-1}Y(x_2, y_2)\phi(t_{12}, t_{13})\exp(-t_{12}/2) \end{aligned} \quad (6.3)$$

$$\begin{aligned} & \phi(t_{12}, t_{23})^{-1}Y(x_1, y_1)\phi(t_{12}, t_{23})\exp(t_{12}/2)\phi(t_{12}, t_{13})^{-1} \cdot \\ & \cdot X(x_2, y_2)\phi(t_{12}, t_{13})\exp(t_{12}/2) = \exp(t_{12}/2)\phi(t_{12}, t_{13})^{-1} \cdot \\ & \cdot X(x_2, y_2)\phi(t_{12}, t_{13})\exp(-t_{12}/2)\phi(t_{12}, t_{23})^{-1}Y(x_1, y_1)\phi(t_{12}, t_{23}) \end{aligned} \quad (6.4)$$

If $X(A, B), Y(A, B)$ is an elliptic associator, then it is easy to see that $\Delta_{\omega_1, \omega_2}^{++}(u_2(X(x_1, y_1)))$ and $\Delta_{\omega_1, \omega_2}^{++}(u_2(Y(x_1, y_1)))$ define an elliptic structure relative to $\mathcal{A}^\partial \rightarrow \mathcal{A}_1^{<p}$.

We will now describe a specific elliptic associator, which was defined in [8] and [13].

Notations: In the completed Lie algebra $\hat{f}(A, B)$, denote:

$$T := [B, A]$$

$$\tilde{A} := \frac{\text{ad}B}{e^{\text{ad}B} - 1}(A) = A - \frac{1}{2}[B, A] + \frac{1}{12}[B, [B, A]] + \dots$$

Note: The coefficients which appear in this expansion are the Bernoulli numbers, which are denoted by B_i .

Definition 6.2. Given a Drinfel'd associator ϕ , let $e(\phi) = (X_\phi, Y_\phi)$ be defined by:

$$X_\phi(A, B) = \phi(\tilde{A}, T) \exp(\tilde{A}) \phi(\tilde{A}, T)^{-1}$$

$$Y_\phi(A, B) = \exp(T/2) \phi(-\tilde{A} - T, T) \exp(B) \phi(\tilde{A}, T)^{-1}$$

Theorem 6.1. $e(\phi)$ is an elliptic associator relative to ϕ .

A proof of this theorem is given in [13] (Proposition 3.8, see also [8] Proposition 5.3). Our goal in this section is to give a different proof of this theorem, based on the following theorem, which relates $e(\phi)$ to the elliptic structure relative to $\mathcal{A}^\partial \rightarrow \mathcal{A}_1^{<p}$ defined in section 4.4 via the LMO functor. Note that $X_\phi(x_1, y_1)$ and $Y_\phi(x_1, y_1)$ both belong to $RU\hat{\mathfrak{t}}_{1,2}$.

Theorem 6.2.

$$\tilde{u}_2(X_\phi(x_1, y_1)) = LMO^<(X_{+,+}) \in \mathcal{A}_1^{<p}(\uparrow^2)/H_2$$

$$\tilde{u}_2(Y_\phi(x_1, y_1)) = LMO^<(Y_{+,+}) \in \mathcal{A}_1^{<p}(\uparrow^2)/H_2$$

Theorem 6.2 would not hold if we replace \tilde{u}_2 by u_2 (see Remark 6.2 below). Therefore, the elliptic structure relative to $\mathcal{A}^\partial \rightarrow \mathcal{A}_1^{<p}$ which is induced by the LMO functor is not the same elliptic structure which is induced by $e(\phi)$. This theorem says that among all the elliptic structures that come from elliptic associators, the elliptic structure induced by $e(\phi)$ is, in a sense, the “closest” to the one induced by the LMO functor.

Remark 6.1. Using the same techniques one might be able to define associators for higher genus, by pulling back the value of the LMO functor on the right tangles.

The rest of this section is dedicated to proving theorems 6.1 and 6.2.

6.2 Proof of Theorem 6.2

We begin with several lemmas. Here and in the following proofs we denote by $t_{ij} \in \mathcal{A}(\uparrow^n, S)$ the diagram with a single edge connecting the i strand and the j strand.

Lemma 6.3. Given a word ω of length 3 and words ω_1, ω_2 and ω_3 in $\{+, -\}$, $\Delta_{\omega_1, \omega_2, \omega_3}^\omega \phi(t_{12}, t_{23}) = 1$ (i.e. the empty diagram) in $\mathcal{A}(\uparrow^{|\omega_1|+|\omega_2|+|\omega_3|})$ (as defined in section 3.2).

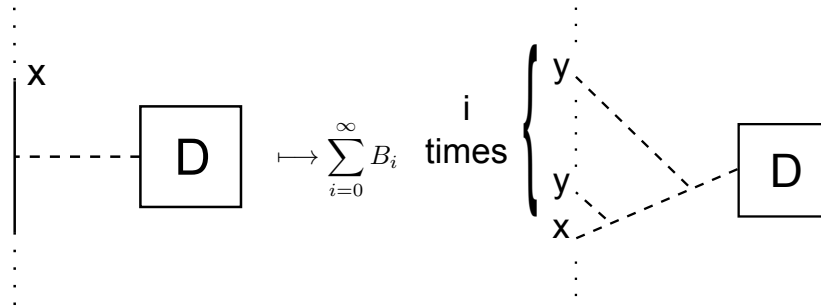
Proof. It is enough to show that $\phi(t_{12}, t_{23}) = 1$ in $\mathcal{A}(\uparrow^3)$. Indeed, using an I relation we get:

$$\phi(t_{12}, t_{23}) = \phi \left(\begin{array}{c} \uparrow \uparrow \uparrow \\ \vdots \vdots \vdots \\ \uparrow \uparrow \uparrow \end{array} \right) = \phi \left(\begin{array}{c} - \uparrow \uparrow \uparrow \\ \vdots \vdots \vdots \\ \uparrow \uparrow \uparrow \end{array} \right) = 1$$

because both $\begin{array}{c} \uparrow \uparrow \uparrow \\ \vdots \vdots \vdots \\ \uparrow \uparrow \uparrow \end{array}$ and $\begin{array}{c} \uparrow \uparrow \uparrow \\ \vdots \vdots \vdots \\ \uparrow \uparrow \uparrow \end{array}$ commute with $\begin{array}{c} \uparrow \uparrow \uparrow \\ \vdots \vdots \vdots \\ \uparrow \uparrow \uparrow \end{array}$ (using the STU relation). \square

Lemma 6.4. Assume that in the pattern \uparrow^{n+1} the left strand is labeled x , and let $a \in \mathcal{A}(\uparrow^{n+1}, \{y\})$. Recall the map $j : \mathcal{A}_1^y \rightarrow \mathcal{A}_1$ defined in section 3.4, and the map $k : \mathcal{A}_1^y \rightarrow \mathcal{A}_1^{<}$ defined in section 3.5. Then we have $\chi_x^{-1} \left(\exp \left(y \cdots \uparrow \cdots \uparrow \right) \cdot a \right) \in \text{Im}(j) \subset^{ts} \mathcal{A}_1(\uparrow^n)$, and $k \circ j^{-1} \circ$

$\chi_x^{-1} \left(\exp \left(y \cdots \uparrow \cdots \uparrow \right) \cdot a \right)$ is the element obtained from a by replacing each vertex on x by the following sum:



where B_i are the Bernoulli numbers.

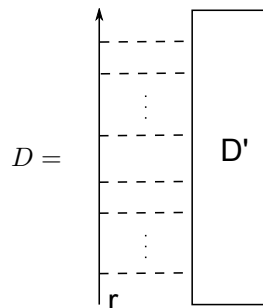
Proof. Recall the element $\lambda(a, b; r) \in \mathcal{A}(\emptyset, \{a, b, r\})$ defined in [10] by:

$$\lambda(a, b; r) = \chi_r^{-1} \left(\begin{array}{c} \exp(a \cdots \uparrow \cdots \uparrow) \\ \exp(b \cdots \uparrow \cdots \uparrow) \end{array} \right) =$$

$$= \exp \left(\begin{array}{c} b \\ \vdots \\ r \end{array} \right) + \sum_{\substack{n \geq 0 \\ u_1, \dots, u_n \in \{a, b\}}} r(u_1, \dots, u_n) \left(\begin{array}{c} u_n \\ \vdots \\ u_1 \\ a \end{array} \right) \right)$$

where $r(u_1, \dots, u_n)$ are some coefficients determined by the Baker-Campbell-Hausdorff formula. In particular we have $r(\underbrace{b, \dots, b}_n) = B_n$ (follows from [12], formula (12)).

$\lambda(a, b; r)$ has the following property: If $D \in \mathbb{D}(P \cup \{\uparrow_r\}, S)$ is of the following form:



then we have:

$$\chi_r^{-1}(D) = \left\langle \chi_{r_1, r_1}^{-1} \left(\begin{array}{c} \uparrow \\ \text{---} \\ \vdots \\ \text{---} \\ \uparrow \\ r_2 \\ \text{---} \\ \vdots \\ \text{---} \\ \uparrow \\ r_1 \end{array} \begin{array}{c} \boxed{D'} \end{array} \right), \lambda(r_1, r_2; r) \right\rangle_{r_1, r_2}$$

where the form $\langle D_1, D_2 \rangle_{r_1, r_2}$ is defined on diagrams D_1, D_2 to be the sum of all ways to glue all vertices labeled by r_1 in D_1 to all vertices labeled r_1 in D_2 and all vertices labeled by r_2 in D_1 to all vertices labeled r_2 in D_2 .

Now, it is enough to prove the theorem for a which is a diagram. Suppose we have $m \geq 1$ vertices on x (the case $m = 0$ is trivial), and denote $x_{(m)} = x$.

$$\begin{aligned} & \chi_{x_{(m)}}^{-1} \left(\begin{array}{c} \text{exp}(y \text{---}) \\ \text{---} \\ \vdots \\ \text{---} \\ \uparrow \\ x_{(m)} \end{array} \begin{array}{c} \boxed{D'} \end{array} \right) = \\ &= \left\langle \chi_{x_m, x_{(m-1)}}^{-1} \left(\begin{array}{c} \text{exp}(y \text{---}) \\ \text{---} \\ \vdots \\ \text{---} \\ \uparrow \\ x_{(m-1)} \\ \text{---} \\ \vdots \\ \text{---} \\ \uparrow \\ x_m \end{array} \begin{array}{c} \boxed{D'} \end{array} \right), \lambda(x_m, x_{(m-1)}; x_{(m)}) \right\rangle_{x_m, x_{(m-1)}} = \\ &= \psi_{x_{(m-1)}, \tilde{x}_{(m-1)}} \left(\sum_{k_m \geq 0} B_{k_m} \cdot \chi_{x_{(m-1)}}^{-1} \left(\text{exp} \left(\begin{array}{c} \tilde{x}_{(m-1)} \\ \vdots \\ x_{(m)} \end{array} \right) \cdot \begin{array}{c} \text{exp}(y \text{---}) \\ \text{---} \\ \vdots \\ \text{---} \\ \uparrow \\ x_{(m-1)} \end{array} \begin{array}{c} \boxed{D'} \end{array} \right) \right) \end{aligned}$$

$k_m \text{ times } \left\{ \begin{array}{c} \tilde{x}_{(m-1)} \\ \vdots \\ \tilde{x}_{(m-1)} \\ x_{(m)} \end{array} \right\}$

where $\psi_{z, \tilde{z}}(D)$ for a diagram D which contains the labels z and \tilde{z} is defined to be the sum of all ways to glue all vertices labeled z to all vertices labeled \tilde{z} .

Repeating this process recursively we get:

$$\begin{aligned}
& \chi_{x(m)}^{-1} \left(\begin{array}{c} \exp(y \text{---}) \\ \vdots \\ \mathbf{x}_{(m)} \end{array} \begin{array}{c} \boxed{\text{D}'} \end{array} \right) = \\
& = \psi_{x(m-1), \tilde{x}(m-1)} \circ \cdots \circ \psi_{x(0), \tilde{x}(0)} \left(\sum_{k_1, \dots, k_m \geq 0} B_{k_1} \cdots B_{k_m} \right. \\
& \quad \left. \exp \left(\begin{array}{c} \tilde{x}_{(m-1)} \\ \vdots \\ x_{(m)} \end{array} \right) \cdots \exp \left(\begin{array}{c} \tilde{x}_{(0)} \\ \vdots \\ x_{(1)} \end{array} \right) \cdot \exp \left(\begin{array}{c} y \\ \vdots \\ x_{(0)} \end{array} \right) \cdot \begin{array}{c} \textcircled{*} \\ \begin{array}{c} k_1 \text{ times} \left\{ \begin{array}{c} \tilde{x}_{(0)} \\ \vdots \\ \tilde{x}_{(0)} \\ x_{(1)} \end{array} \right\} \\ \vdots \\ k_m \text{ times} \left\{ \begin{array}{c} \tilde{x}_{(m-1)} \\ \vdots \\ \tilde{x}_{(m-1)} \\ x_{(m)} \end{array} \right\} \end{array} \right. \\
& \quad \left. \begin{array}{c} \boxed{\text{D}'} \end{array} \right) \left. \right)
\end{aligned}$$

For a specific choice of k_1, \dots, k_m , we wish to describe the element we get by applying $\psi_{x(m-1), \tilde{x}(m-1)} \circ \cdots \circ \psi_{x(0), \tilde{x}(0)}$ to the corresponding summand. A careful examination shows that the element we get is a product of $\exp \left(\begin{array}{c} y \\ \vdots \\ x_{(m)} \end{array} \right)$ with the sum of all the diagrams which can be produced from $\textcircled{*}$ by the following process:

1. Change all the $\tilde{x}_{(0)}$ labels to y .
2. For $i = 1, \dots, m-1$:
 - Attach some of the $x_{(i)}$ labels to some of the $\tilde{x}_{(i)}$ labels.
 - Change all the remaining $x_{(i)}$ labels to $x_{(i+1)}$.
 - Change all the remaining $\tilde{x}_{(i)}$ labels to y .

This sum can be described more shortly as the sum of all ways to glue some of the $x_{(i)}$ labels to $\tilde{x}_{(j)}$ labels with $j \geq i$, and then change all the remaining $x_{(i)}$ labels to $x_{(m)} = x$ and all the remaining $\tilde{x}_{(i)}$ labels to y .

The result of the above calculation clearly belongs to the image of j . It is not difficult to see

that $j^{-1} \circ \chi_x^{-1} \left(\exp \left(y \cdots \uparrow \cdots \uparrow \right) \cdot D \right)$ has a simpler presentation when mapped by k to $\mathcal{A}_1^<(\uparrow^n)$, as:

$$\sum_{k_1, \dots, k_m \geq 0} B_{k_1} \cdots B_{k_m}$$

Thus we have completed the proof of the lemma. □

Lemma 6.5. Recall the notation $\tilde{A} := \frac{adB}{e^{adB}-1}(A)$ in $\hat{f}(A, B)$. Similarly we have in $U\hat{\mathfrak{t}}_{1,2}$: $\tilde{x}_1 = \frac{ady_1}{e^{ady_1}-1}(x_1)$. Then we have:

$$\tilde{u}_2(\tilde{x}_1) \approx \sum_{i=0}^{\infty} B_i \quad i \text{ times } \left\{ \begin{array}{c} y \\ \vdots \\ y \\ x \end{array} \right\} \quad \text{mod } H_2$$

Proof. $\tilde{x}_1 = \sum_{i=0}^{\infty} B_i \underbrace{[y_1[\cdots [y_1, x_1] \cdots]]}_{i \text{ times}}$ by definition. We will prove by induction on $i \geq 1$ the following identity, which will imply the lemma:

$$u_2(\underbrace{[y_1[\cdots [y_1, x_1] \cdots]]}_{i \text{ times}}) = i \text{ times } \left\{ \begin{array}{c} y \\ \vdots \\ y \\ x \end{array} \right\} + (i-1) \text{ times } \left\{ \begin{array}{c} y \\ \vdots \\ y \end{array} \right\}$$

For $i = 1$ we have:

$$u_2([y_1, x_1]) = \begin{array}{c} y \text{ ---} \uparrow \\ x \text{ ---} \uparrow \end{array} - \begin{array}{c} x \text{ ---} \uparrow \\ y \text{ ---} \uparrow \end{array} = \begin{array}{c} y \text{ ---} \uparrow \\ x \text{ ---} \uparrow \end{array} -$$

$$\begin{aligned}
& - \begin{array}{c} y \\ \diagdown \\ x \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} + \begin{array}{c} y \\ \diagup \\ x \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} - \begin{array}{c} x \\ \text{---} \\ y \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} = \\
& = \begin{array}{c} y \\ \diagdown \\ x \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} + \begin{array}{c} \uparrow \\ \uparrow \end{array}
\end{aligned}$$

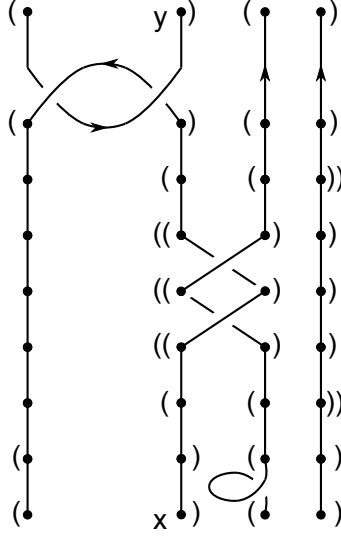
Assume we proved the lemma for i_0 . Then for $i_0 + 1$:

$$\begin{aligned}
u_2(\underbrace{[y_1[\cdots [y_1, x_1] \cdots]]}_{i_0 + 1 \text{ times}}) &= \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ y \\ \diagup \\ x \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} - \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ x \\ \diagup \\ y \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} + \\
& \overbrace{\begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ y \\ \diagup \\ y \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} - \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ y \\ \diagup \\ y \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}}^{=0} = \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ y \\ \diagup \\ x \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} - \\
& - \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ y \\ \diagup \\ x \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} + \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ y \\ \diagup \\ x \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} - \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ x \\ \diagup \\ y \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} = \\
& = \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ y \\ \diagup \\ x \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array} + \begin{array}{c} y \\ \text{---} \\ y \\ \diagdown \\ y \\ \diagup \\ y \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}
\end{aligned}$$

□

We are now ready to prove theorem 6.2.

Proof of theorem 6.2. We represent the tangle $X_{+,+}$ as follows:



We will calculate $\chi_x(LMO(X_{+,+})) \in \mathcal{A}(\uparrow_x \uparrow^2, \{y\})$. Inside $\mathcal{A}(\uparrow_x \uparrow^2, \{y\})$ we have the subspace H_2 which is spanned by all diagrams with a component which has more than one vertex on the second strand from the right, or has a loop. For an element $a \in H_2 \subset \mathcal{A}(\uparrow_x \uparrow^2, \{y\})$ which is mapped by χ_x^{-1} to the image of $j : \mathcal{A}_1^{yp} \rightarrow \mathcal{A}_1^p$, we have $k \circ j^{-1} \circ \chi_x^{-1}(a) \in H_2 \subset \mathcal{A}_1^{<p}(\uparrow^n)$.

$$\begin{aligned}
& \chi_x(T_1(++)) \circ \chi_y^{-1} \circ Z \left(\begin{array}{cc} \begin{array}{c} \text{Diagram 1} \end{array} & \begin{array}{c} \text{Diagram 2} \end{array} \end{array} \right) = \\
& = \begin{array}{c} \text{exp} \left(\begin{array}{c} \text{Diagram 3} \end{array} \right) \\ \text{Diagram 4} \end{array} \\
& Z \left(\begin{array}{cc} \begin{array}{c} \text{Diagram 5} \end{array} & \begin{array}{c} \text{Diagram 6} \end{array} \end{array} \right) = 1 \quad \text{by lemma 6.3} \\
& Z \left(\begin{array}{cc} \begin{array}{c} \text{Diagram 7} \end{array} & \begin{array}{c} \text{Diagram 8} \end{array} \end{array} \right) = \phi(t_{23}, t_{34}) \\
& Z \left(\begin{array}{cc} \begin{array}{c} \text{Diagram 9} \end{array} & \begin{array}{c} \text{Diagram 10} \end{array} \end{array} \right) = \exp(t_{23}) \\
& Z \left(\begin{array}{cc} \begin{array}{c} \text{Diagram 11} \end{array} & \begin{array}{c} \text{Diagram 12} \end{array} \end{array} \right) = \phi(t_{23}, t_{34})^{-1} \\
& Z \left(\begin{array}{cc} \begin{array}{c} \text{Diagram 13} \end{array} & \begin{array}{c} \text{Diagram 14} \end{array} \end{array} \right) = 1 \quad \text{by lemma 6.3}
\end{aligned}$$

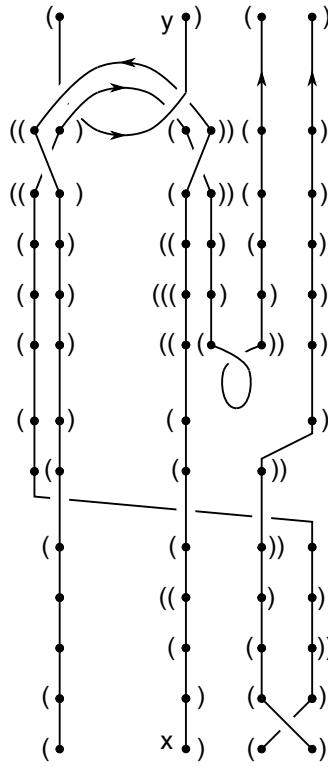
$$Z \left(\begin{array}{ccc} \bullet & \bullet & \bullet \\ \downarrow & \updownarrow & \downarrow \\ \bullet & \bullet & \bullet \end{array} \right) \approx 1 \pmod{H_2}$$

Putting it all together we get only 3 strands, since the 2 left strands are connected at the top, and become one strand labeled by x . Note that in all the the above diagrams we have no vertex on the left strand, therefore after the composition the y -labeled edge is at the top of the x strand. Hence we get the following element of $\mathcal{A}(\uparrow_x \uparrow^2, \{y\})$:

$$\chi_x(LMO(X_{+,+})) \approx \exp \left(\begin{array}{c} y \cdots \cdots \\ \vdots \\ x \end{array} \middle| \begin{array}{c} | \\ | \\ | \end{array} \right) \cdot \phi(t_{12}, t_{23}) \cdot \exp(t_{12}) \cdot \phi(t_{12}, t_{23})^{-1} \mod H_2$$

The proof of the theorem for $X_{+,+}$ is now completed by Lemmas 6.4 and 6.5.

For $Y_{+,+}$ we find it easier to carry out the calculation on $Y_{+,+}^{-1}$. We use the following presentation:

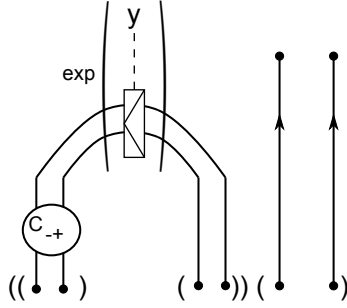


In the following calculation, note that in some of the slices we get elements which by themselves are not equivalent to $1 \pmod{H_2}$, but they become equivalent to 1 after composing all the diagrams.

First we need to calcualte:

$$\chi_x(T_1(++)) \circ \chi_y^{-1} \circ Z \left(\begin{array}{cc} \bullet & \bullet \\ \downarrow & \downarrow \\ \bullet & \bullet \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{cc} \bullet & \bullet \\ \downarrow & \downarrow \\ \bullet & \bullet \end{array} \right)$$

According to Lemma 5.5 of [10], this can be computed as:



where C_{-+} is $Z(\text{diagram}) \in \mathcal{A}(\text{diagram}) \cong \mathcal{A}(\uparrow\uparrow)$. We will show at the end of the proof that $C_{++} \equiv 1 \pmod{H_2}$. Therefore:

$$\begin{aligned} & \chi_x(T_1(++)) \circ \chi_y^{-1} \circ Z \left(\text{diagram} \right) = \\ & = \exp \left(\text{diagram} \right) \exp \left(- \text{diagram} \right) \\ & Z \left(\text{diagram} \right) = \exp(-t_{12}/2) \exp(t_{34}/2). \end{aligned}$$

Since those exponents commute with the box coming from the above tangle, they cancel each other and do not contribute to the final expression.

$$\begin{aligned} & Z \left(\text{diagram} \right) = 1 \quad \text{by lemma 6.3} \\ & Z \left(\text{diagram} \right) = \phi(t_{35} - t_{45}, t_{56}) \approx \\ & \approx \phi(t_{35}, t_{56}) \pmod{H_2} \quad (\text{after composition}) \\ & Z \left(\text{diagram} \right) = \phi(-t_{34}, -t_{45})^{-1} \approx 1 \pmod{H_2} \quad (\text{after composition}) \\ & Z \left(\text{diagram} \right) \approx 1 \pmod{H_2} \\ & Z \left(\text{diagram} \right) = 1 \quad \text{by lemma 6.3} \\ & Z \left(\text{diagram} \right) = \exp \left(\frac{1}{2} \text{diagram} \right) = \quad (\text{using an } I \text{ relation}) \\ & = \exp \left(-\frac{1}{2} \text{diagram} \right) \approx 1 \pmod{H_2} \\ & Z \left(\text{diagram} \right) = 1 \quad \text{by lemma 6.3} \\ & Z \left(\text{diagram} \right) = \phi(t_{23}, t_{34})^{-1} = \phi(-t_{24} - t_{34}, t_{34})^{-1} \end{aligned}$$

(because $t_{23} + t_{24} + t_{34}$ commutes with t_{23} and t_{34})

$$Z \left(\begin{array}{ccc} \downarrow & \uparrow & \uparrow \uparrow \\ \bullet & \bullet & \bullet \bullet \\ \downarrow & \uparrow & \uparrow \uparrow \\ \bullet & \bullet & \bullet \bullet \end{array} \right) = 1 \quad \text{by lemma 6.3}$$

$$Z \left(\begin{array}{ccc} \downarrow & \uparrow & \uparrow \uparrow \\ \bullet & \bullet & \bullet \bullet \\ \downarrow & \uparrow & \uparrow \uparrow \\ \bullet & \bullet & \bullet \bullet \end{array} \right) = \exp(-t_{34}/2)$$

Putting it all together we get the following element of $\mathcal{A}(\uparrow_x \uparrow^2, \{y\})$:

$$\chi_x(LMO(Y_{+,+}^{-1})) \approx \exp \left(\begin{array}{c} y \cdots \downarrow \\ \vdots \\ x \end{array} \left| \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right. \right).$$

$$\cdot \phi(t_{12}, t_{23}) \exp \left(- \begin{array}{c} y \cdots \downarrow \\ \vdots \\ x \end{array} \left| \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right. \right) \phi(-t_{12} - t_{23}, t_{23})^{-1} \exp(-t_{23}/2) \quad \text{mod } H_2$$

Note that when we travel along strand x we encounter $\exp \left(\begin{array}{c} y \cdots \downarrow \\ \vdots \\ x \end{array} \left| \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right. \right)$ after the associator $\phi(t_{12}, t_{23})$, whereas when we travel along the second strand we encounter $\exp \left(- \begin{array}{c} y \cdots \downarrow \\ \vdots \\ x \end{array} \left| \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right. \right)$ before this associator.


By Lemmas 6.4 and 6.5 we get:

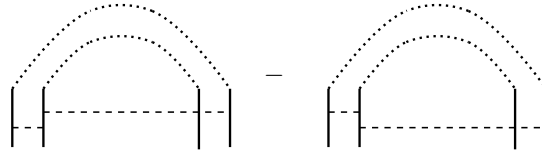
$$LMO^<(Y_{+,+}^{-1}) = \tilde{u}_2(\phi(\tilde{x}_1, t) \exp(-y_1) \phi(-\tilde{x}_1 - t, t)^{-1} \exp(-t/2))$$

Therefore:

$$LMO^<(Y_{+,+}) = \tilde{u}_2(\exp(t/2) \phi(-\tilde{x}_1 - t, t) \exp(y_1) \phi(\tilde{x}_1, t)^{-1})$$

which is the expression we wanted to get.

In order to complete the proof we only need to show that $C_{-+} \approx 1 \text{ mod } H_2$. C_{-+} is the composition of  with $\phi(-t_{23}, -t_{34}) \phi(-t_{12}, -t_{23} + t_{24}) \in \mathcal{A}(\downarrow \uparrow \downarrow \uparrow)$. t_{23} is in H_2 (after the composition), therefore we are left with $\phi(-t_{12}, t_{24})$. This can be written as the exponent of a sum of iterated commutators, where the innermost commutator is:



Using several STU and IHX relations we can “transfer” the nodes on the rightmost strand to nodes on the leftmost strand, in the price of adding many more diagrams in which this node is transferred to the lower capped strand. But all those extra diagrams are in H_2 . Therefore we are left with the commutator:



which, by another STU relation (on the left strand), is also in H_2 . Thus we have completed the proof. \square

Remark 6.2. Looking at the above calculation we see why the H relations were needed. For example, in $LMO^<(X_{+,+}) \in \mathcal{A}_1^{<p}(\uparrow^2)$ we have all the high degree components of $Z \left(\begin{array}{c} \text{diagram} \end{array} \right)$ which do not vanish (with the exception of the 2 degree component which vanishes, see the proof of Theorem 6.1 in the next section), and are not in the image of the map u_2 , so they do not come from the elliptic associator $e(\phi)$.

6.3 Proof of Theorem 6.1

In this section we use the relation we found between $e(\phi)$ and the LMO functor to give an alternative proof that $e(\phi)$ is an elliptic associator. In fact, most of the proof follows from the fact that $LMO^<(X_{+,+})$ and $LMO^<(Y_{+,+})$ define an elliptic structure on $\mathcal{A}_1^{<p}(\uparrow^n)$.

Indeed, identities (2.3) and (2.4) hold for $X_{+,+}$ and $Y_{+,+}$ in $q\tilde{T}_1$. Applying $LMO^<$ to both sides and taking the quotient with H_2 (for 2.3) and H_3 (for 2.4) we get the same identities for $LMO^<(X_{+,+})$ and $LMO^<(Y_{+,+})$ in $R\mathcal{A}_1^{<p}(\uparrow^2)/H_2$ and $R\mathcal{A}_1^{<p}(\uparrow^3)/H_3$. Pulling them back to $RU\hat{\mathbf{t}}_{1,2} \subset U\hat{\mathbf{t}}_{1,2}$ and $RU\hat{\mathbf{t}}_{1,3} \subset U\hat{\mathbf{t}}_{1,3}$ by $\tilde{u}_{r,2}^{-1}$ and $\tilde{u}_{r,3}^{-1}$, and using theorem 6.2, we get identities (6.1) and (6.4) for X_ϕ and Y_ϕ .

Similarly, identities (2.1) and (2.2) hold for $X_{+,+}$ and $Y_{+,+}$ in $q\tilde{T}_1$. We can now again apply $LMO^<$ to both sides, take the quotient with H_3 and pull back to $RU\hat{\mathbf{t}}_{1,3} \subset U\hat{\mathbf{t}}_{1,3}$ by $\tilde{u}_{r,3}^{-1}$. Theorem 6.2 shows that at the right hand sides we get the right hand sides of identities (6.2) and (6.3) for X_ϕ and Y_ϕ . However, we still need to show that the left hand sides are equal. More explicitly, we need to show that $\tilde{u}_{r,3}^{-1} \circ LMO^<(Z_{+,+}) = Z_\phi(x_1 + x_2, y_1 + y_2)$ for $Z = X$ and $Z = Y$. This will be the content of the following proof.

Proof of theorem 6.1. Let $\Delta_1 : RU\hat{\mathbf{t}}_{1,2} \rightarrow RU\hat{\mathbf{t}}_{1,3}$ be the map defined by : $v_1 \mapsto v_1 + v_2$ and $v_2 \mapsto v_3$ ($v = x$ or $v = y$). As explained above, we need to calculate $\tilde{u}_{r,3}^{-1} \circ LMO^<(Z_{+,+})$ for $Z = X, Y$, and show that they are equal to $\Delta_1 \circ \tilde{u}_{r,2}^{-1} LMO^<(Z_{+,+})$. We will use the same decomposition of $X_{+,+}$ and $Y_{+,+}$ to simple tangles as we used above. For most of those simple tangles T we can show that:

$$\tilde{u}_{r,3}^{-1} \circ LMO^<(\Delta_{+,+,+}^{++}(T)) = \Delta_1 \circ \tilde{u}_{r,2}^{-1}(LMO^<(T)) \quad (6.5)$$


In the few cases where this identity does not hold, the extra components we get will eventually cancel each other.

For a tangle T with no cups or caps, we can use the identity $LMO^<(\Delta_{+,+,+}^{++}(T)) = \Delta_{+,+,+}^{++} \circ LMO^<(T)$. Decompose $LMO^<(T)$ as $LMO^<(T)_u + LMO^<(T)_H$, where $LMO^<(T)_u$ is in the image of u_2 , and $LMO^<(T)_H$ is in H_2 . Clearly, $\tilde{u}_{r,3}^{-1} \circ \Delta_{+,+,+}^{++}(LMO^<(T)_u) = \Delta_1 \circ \tilde{u}_{r,2}^{-1}(LMO^<(T)_u)$. Therefore, in order to prove identity (6.5) for such T , it is enough to show that $\Delta_{+,+,+}^{++}(LMO^<(T)_H)$ is in H_3 .


In the calculation of the $LMO^<$ functor of $X_{+,+}$ and $Y_{+,+}$ we encountered several contributions to the H_2 part. First, according to lemma 6.4, each edge with a vertex on the x strand, after

applying $k \circ j^{-1} \circ \chi_x^{-1}$, became:

$$\sum_{i=0}^{\infty} B_i \quad i \text{ times } \left\{ \begin{array}{l} y \\ y \\ x \end{array} \right. \rightarrow \boxed{D} \quad (6.6)$$

The H_2 part of this sum is, according to lemma 6.5: $\sum_{i=1}^{\infty} B_i$  . For i even we have:

$$\Delta_{+++,+}^{++} \left(\begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} \right) = \begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} + \begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} + \begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} + \begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} + \begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} + \begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} = \begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} + \begin{array}{c} \text{i-1} \\ \text{times} \left\{ \begin{array}{c} y \\ y \end{array} \right\} \end{array} \in H_3$$

For i odd, the only non-zero coefficient in the sum (6.6) is $B_1 = -\frac{1}{2}$. When this sum appears in an associator, the $-\frac{1}{2}$  summand cancels, because it commutes with everything else.

In $LMO^<(X_{+,+})$ the sum (6.6) appears twice inside an associator, so the H_2 part of those tangle-parts indeed maps to H_3 . There is also one occurrence of this sum which appears inside an exponent. Therefore we are left with $\exp(-\frac{1}{2} \text{ (diagram) })$ which is in H_2 , but is not mapped by $\Delta_{+,+,+}^{++}$ to H_3 . However, we will immediately see that this part eventually cancels.

In $LMO^{<}(X_{+,+})$ we also have $Z\left(\begin{array}{c} \text{cup} \\ \text{cap} \end{array}\right)$ which has a cup and a cap. We need to calculate $Z\left(\Delta_{++,+}^{++}\left(\begin{array}{c} \text{cup} \\ \text{cap} \end{array}\right)\right) = Z\left(\begin{array}{c} \text{link} \end{array}\right)$. This can be written as $\exp(t_{34} + u)$, where u is in H_3 . So we are left with $\exp(t_{34})$. But this part cancels with $\exp(-t_{34})$ coming from

applying $\Delta_{++,+}^{++}$ to the exponent of the sum (6.6) (because they commute with everything in between). This concludes the proof for $LMO^<(X_{+,+})$.

In $LMO^<(Y_{+,+})$, all the occurrences of the sum (6.6) are in associators, so their H_2 part is mapped to H_3 . But there are several more contributions to the H_2 part. First, we had:

$$Z \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = \exp \left(-\frac{1}{2} \begin{array}{c} \text{Diagram 2} \end{array} \right)$$

Applying $\Delta_{++,+}^{++}$ to $\exp \left(-\frac{1}{2} \begin{array}{c} \text{Diagram 2} \end{array} \right)$ we get $\exp(-t_{56})$ which is not in H_3 . However, we will soon see that this exponent cancels with another exponent.

Another contribution to the H_2 part comes from $Z \left(\begin{array}{c} \text{Diagram 3} \end{array} \right)$. So we need to calculate $Z(\Delta_{++,+}^{++} \left(\begin{array}{c} \text{Diagram 4} \end{array} \right)) = Z \left(\begin{array}{c} \text{Diagram 5} \end{array} \right)$. This is equal to $\exp(t_{45} + u)$ with $u \in H_3$. So we are left with $\exp(t_{45})$ which is not in H_3 . But this cancels out with the above $\exp(-t_{56})$ (because they commute with everything in between).

We will now deal with the H_2 parts which come from: $Z \left(\begin{array}{c} \text{Diagram 6} \end{array} \right)$ and $Z \left(\begin{array}{c} \text{Diagram 7} \end{array} \right)$, which are:

$$\phi \left(\begin{array}{c} \text{Diagram 8} \end{array} \right) - \begin{array}{c} \text{Diagram 9} \end{array}, \quad \phi^{-1} \left(\begin{array}{c} \text{Diagram 10} \end{array} \right) \cdot \begin{array}{c} \text{Diagram 11} \end{array}$$

(For convenience we ignored the 2 left strands.)

$\phi^{-1} \left(\begin{array}{c} \text{Diagram 12} \end{array} \right)$ is an exponent of a sum of iterated commutators in $\begin{array}{c} \text{Diagram 13} \end{array}$ and $\begin{array}{c} \text{Diagram 14} \end{array}$. The innermost commutator in each of those iterated commutators is $\left[\begin{array}{c} \text{Diagram 15} \end{array}, \begin{array}{c} \text{Diagram 16} \end{array} \right]$, so it is enough to show that applying $\Delta_{++,+}^{++}$ to this commutator multiplied by $\begin{array}{c} \text{Diagram 17} \end{array}$ is in H_3 . (Note that we apply $\Delta_{++,+}^{++}$ to the right strands. The left strand will disappear when we apply χ_x^{-1} .) And indeed:

$$\Delta_{++,+}^{++} \left(\begin{array}{c} \text{Diagram 18} \end{array} - \begin{array}{c} \text{Diagram 19} \end{array} \right) =$$

We are left with summands in which the empty diagram of:

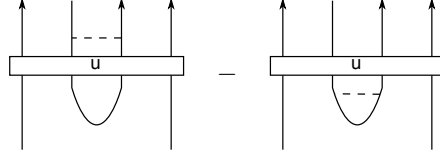
$$\phi^{-1}\left(\begin{array}{ccccc} \uparrow & \downarrow & \uparrow & \uparrow & \\ | & \cdots & | & | & \\ \downarrow & & \downarrow & \uparrow & \end{array}, \begin{array}{ccccc} \uparrow & \downarrow & \uparrow & \uparrow & \\ | & \cdots & | & | & \\ \downarrow & & \downarrow & \uparrow & \end{array}\right) \cdot \left(\begin{array}{ccccc} \uparrow & \cup & \uparrow & \uparrow & \\ | & & | & | & \end{array}\right)$$

is multiplied by the H_2 part of:

[illegible]

This associator is an exponent of a sum of iterated commutators of $\begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}$, $\begin{array}{c} \uparrow \\ \uparrow \\ \text{---} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}$ and $\begin{array}{c} \uparrow \\ \downarrow \\ \text{---} \\ \uparrow \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}$. If $\begin{array}{c} \uparrow \\ \downarrow \\ \text{---} \\ \uparrow \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}$ does not appear in this commutator, it does not contribute to the H_2 part. So we may consider only commutators in which $\begin{array}{c} \uparrow \\ \uparrow \\ \text{---} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}$ appears. We may assume the commutator is one sided. It is also enough to consider commutators of the type $\left[\begin{array}{c} \uparrow \\ \uparrow \\ \text{---} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}, u \right]$ where u is a commutator in $\begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}$ and $\begin{array}{c} \uparrow \\ \downarrow \\ \text{---} \\ \uparrow \end{array} \begin{array}{c} \uparrow \\ \uparrow \end{array}$ (because all the commutators we consider here have this type of commutator as their inner part). So we

need to consider elements of the following form:



where u has no vertices on the down-going strand \downarrow . It is easy to prove (by induction) that by applying Δ to the strands 2 and 3 of u we get $u_1 + u_2$, where u_1 is obtained by putting all the vertices of strand 3 (from the left) in $\uparrow\downarrow\uparrow\uparrow$ on strand 4 of $\uparrow\downarrow\uparrow\uparrow$, and u_2 is obtained by putting all the vertices of strand 3 in $\uparrow\downarrow\uparrow$ on strand 5 of $\uparrow\downarrow\uparrow\uparrow$. So we have:

$$\begin{aligned}
& \Delta_{++,+}^{++} \left(\text{Diagram 1} - \text{Diagram 2} \right) = \\
& = \underbrace{\text{Diagram 3} - \text{Diagram 4} + \text{Diagram 5} - \text{Diagram 6}}_{\in H_3} + \\
& + \underbrace{\text{Diagram 7} - \text{Diagram 8}}_{\text{canceling}} + \underbrace{\text{Diagram 9} - \text{Diagram 10}}_{\text{canceling}} + \\
& + \text{Diagram 11} - \text{Diagram 12} + \text{Diagram 13} - \text{Diagram 14}
\end{aligned}$$

u_1 and u_2 can be written as a sum of connected diagrams. The diagrams in this sum which have more than 1 vertex on strand 4 (for u_1) or on strand 5 (for u_2) are already in H_3 . For diagrams with only one vertex on those strands, the above sum equals 0 via the STU relation.

At last we have to deal with the H_2 part coming from C_{-+} . Most of the summands in C_{-+} are mapped to H_3 by the exact same argument we have just seen. We only need to show that

$$\Delta_{++,+}^{++} \left(\text{Diagram 15} \circ \phi(-t_{12}, t_{24}) \right) \text{ is in } H_3.$$

$\phi(-t_{12}, t_{24})$ is an exponent of a linear combination of commutators in t_{12} and t_{24} . In each summand of the exponent, one of the commutators is closest to the caps at the top. This commutator can be written as a one-component diagram with only one vertex on strand 2. So after applying $\Delta_{++,+}^{++}$ we get two copies of this diagram, each with a vertex on each copy of strand 2. The same argument from the end of the proof of theorem 6.2, which showed that C_{-+} is in H_2 , shows now

that each of these copies is in H_3 . This completes the proof for $LMO^<(Y_{+,+})$, and hence the proof of theorem 6.1. □

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